Capacity Bounds via Productization

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Outline

1. Introduction and Main Result
2. Applications
   - Matrix Scaling Bound
   - Metric TSP
3. Proof
   - Bound for Product Polynomials
   - Productization of Real Stable Polynomials
4. Open Questions

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Polynomial Capacity

Given $p \in \mathbb{R}[x_1, \ldots, x_n]$ with non-negative coefficients:

$$\mathrm{Cap}_{\mathbf{1}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\mathbf{1}}} = \inf_{x_1,\ldots,x_n>0} \frac{p(x_1,\ldots,x_n)}{x_1 x_2 \cdots x_n}.$$

Applications include bounds/approximations for:

- Permanent and mixed discriminant (Gurvits)
- Contingency tables and bipartite matchings (Barvinok, Barvinok-Hartigan, Gurvits, Gurvits-L, Brändén-L-Pak)
- Eulerian orientations (Csikvári-Schweitzer)
- Counting/optimization on stable matroids (Straszak-Vishnoi, Anari-Oveis Gharan)
- Intersection of two general matroids (Anari-Oveis Gharan-Vinzant)
- Operator scaling and invariant theory (combinations of Bürgisser, Franks, Garg, Gurvits, Oliveira, Walter, Wigderson)

Real Stable and Log-concave Polynomials

Almost all applications via real stable and log-concave polynomials.

A polynomial p is **real stable** if

$$p(z_1, \ldots, z_n) \neq 0 \qquad \text{whenever} \qquad \mathrm{Im}(z_i) > 0 \quad \text{for all } i.$$

- Newton's inequalities for coefficients of real-rooted polynomials generalized by strong Rayleigh inequalities (Brändén).
- Log-concave in the positive orthant $= \mathbb{R}^n_+$.

A polynomial p is **strongly log-concave** (Gurvits) if

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} p \quad \text{is log-concave in } \mathbb{R}^n_+ \qquad \forall \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n_+.$$

- Connects matroids and the Alexandrov-Fenchel inequalities.
- Also called **completely log-concave** (Anari-Liu-Oveis Gharan-Vinzant) and **Lorentzian** (Brändén-Huh).

Probabilistic Interpretation

Given $p \in \mathbb{R}[x_1, \ldots, x_n]$ with non-negative coefficients:

$$\mathrm{Cap}_{\mathbf{1}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\mathbf{1}}} = \inf_{x_1,\ldots,x_n>0} \frac{p(x_1,\ldots,x_n)}{x_1 x_2 \cdots x_n}.$$

Let $p(\mathbf{1}) = 1$ and consider probability distribution $\boldsymbol{\mu}$ on $\mathrm{supp}(p) \subset \mathbb{Z}^n$:

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{\kappa}} p_{\boldsymbol{\kappa}} \boldsymbol{x}^{\boldsymbol{\kappa}} \iff \mathbb{P}[\boldsymbol{\mu} = \boldsymbol{\kappa}] = p_{\boldsymbol{\kappa}},$$

$$\nabla p(\mathbf{1}) = \mathbb{E}[\boldsymbol{\mu}] \quad (\text{“marginals”})$$

We have $0 \leq \mathrm{Cap}_{\mathbf{1}}(p) \leq 1$ and:

- $\mathrm{Cap}_{\mathbf{1}}(p) > 0$ iff $\mathbf{1}$ is in the Newton polytope of $p = \mathrm{hull}(\mathrm{supp}(p))$.
- $\mathrm{Cap}_{\mathbf{1}}(p) = 1$ iff marginals $= \mathbf{1}$ (p is **doubly stochastic**).

Main Result

What if the marginals are only <u>close</u> to 1? (think algos)

Theorem (Gurvits-L '20)

Let p be an n-variate homogeneous polynomial of degree n with $p(\mathbf{1}) = 1$. If p is real stable and $\|\mathbf{1} - \nabla p(\mathbf{1})\|_1 < 2$, then

$$1 \geq \mathrm{Cap}_{\mathbf{1}}(p) = \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\mathbf{1}}} \geq \left(1 - \frac{\|\mathbf{1} - \nabla p(\mathbf{1})\|_1}{2}\right)^n.$$

Use Gurvits' original coefficient-capacity bound to get:

Corollary

If p and $\nabla p(\mathbf{1})$ are as in the previous theorem, then

$$p_{\mathbf{1}} \geq \frac{n!}{n^n} \cdot \mathrm{Cap}_{\mathbf{1}}(p) \geq \frac{n!}{n^n}\left(1 - \frac{\|\mathbf{1} - \nabla p(\mathbf{1})\|_1}{2}\right)^n.$$

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Matrix Scaling: Setup

Goal: Given matrix A with non-negative entries, want to multiply on left and right by diagonal matrices to make A doubly stochastic ("**scaling**").

Idea: Normalize rows, then columns, then rows, then columns, ...

Linial-Samorodnitsky-Wigderson '00: We can use this method to deterministically approximate the permanent within e^n factor.

1. Easy to keep track of changes to permanent (det of diagonal matrix).
2. Have an e^n approximation of doubly stochastic permanent:

$$1 \geq \mathrm{per}(A) \geq \frac{n!}{n^n} \quad (\longleftarrow \textbf{vdW bound: } \text{Egorychev, Falikman}).$$

3. Need a similar bound when a matrix is "close" to doubly stochastic:

$$\|\mathbf{1} - \boldsymbol{c}\|_2 < \frac{1}{\sqrt{n}} \implies \mathrm{per}(M) \geq \frac{n!}{n^n}\left(1 - \sqrt{n}\|\mathbf{1} - \boldsymbol{c}\|_2\right)^n,$$

where $\boldsymbol{c}$ are the column sums of A and rows sums are $\mathbf{1}$.

Matrix Scaling: Our Bound

To use our bound on per(A)**:**

$$p(\boldsymbol{x}) := \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j \implies \text{per}(A) = p_{\mathbf{1}},$$

$$\text{row sums} = \mathbf{1} \implies p(\mathbf{1}) = 1 \qquad \text{and} \qquad \text{column sums} = \nabla p(\mathbf{1}).$$

Using our bound: $\|\mathbf{1} - \nabla p(\mathbf{1})\|_2 < \frac{2}{\sqrt{n}}$ implies

$$\text{per}(A) \geq \frac{n!}{n^n} \left(1 - \frac{\|\mathbf{1} - \nabla p(\mathbf{1})\|_1}{2}\right)^n \geq \frac{n!}{n^n} \left(1 - \frac{\sqrt{n}\|\mathbf{1} - \nabla p(\mathbf{1})\|_2}{2}\right)^n.$$

Similar inequalities used for more recent **operator/tensor scaling**.

Metric TSP: Introduction

Metric TSP: "travelling salesperson problem" assuming the triangle inequality (NP-hard, even approximation with factor $\frac{123}{122}$).

Christofides-Serdyukov ('70s): approximation for TSP with factor $\frac{3}{2}$.

- Min spanning tree + min matching on odd degree vertices of tree.

Karlin-Klein-Oveis Gharan ('20): factor improved to $\approx \frac{3}{2} - 10^{-36}$.

- **Instead:** Random spanning tree based on linear relaxation.

Significant improvements for many special cases, but this is the first general improvement.

We now discuss connections to our bound. For more discussion about the algorithm itself, see the talk on Friday by Nathan Klein.

Metric TSP: Setup

Let $\boldsymbol{\nu}$ be a distribution on $2^{[m]}$ (random subset of $[m]$), and associate to $\boldsymbol{\nu}$ the (non-homogeneous) polynomial q, as above:

$$q(\boldsymbol{y}) := \sum_{S \subseteq [m]} \mathbb{P}[\boldsymbol{\nu} = S] \boldsymbol{y}^S.$$

E.g.: uniform distribution on spanning trees $\implies q(\boldsymbol{y})$ real stable.
E.g.: spanning tree T sampled according to $\boldsymbol{\lambda}^T \implies q(\boldsymbol{y})$ real stable.

Given disjoint sets $S_1 \sqcup \cdots \sqcup S_n = [m]$, construct random variables:

$$A_i := \sum_{j \in S_i} \nu_j, \qquad \forall i \in [n].$$

That is: if $X \subseteq [m]$ is drawn from $\boldsymbol{\nu}$, then $A_i = |X \cap S_i|$.

Metric TSP: Translation

Their paper: "Roughly speaking, (when q is real stable) we show that

$$\|\mathbf{1} - \mathbb{E}[\boldsymbol{A}]\|_1 < 1 - \epsilon \quad \implies \quad \mathbb{P}[\boldsymbol{A} = \mathbf{1}] \geq f(\epsilon, n),$$

where $f(\epsilon, n) \sim \epsilon^{2^n}$ has no dependence on m."

Now consider: $\tilde{p}(\boldsymbol{x}) := q(\boldsymbol{y})\big|_{y_j = x_i \text{ for } j \in S_i}$

- $\tilde{p}(\boldsymbol{x})$ has n variables, and $\deg(\tilde{p}) = \deg(q)$.
- $\tilde{p}(\mathbf{1}) = 1$ and marginals $= \nabla\tilde{p}(\mathbf{1}) = \mathbb{E}[\boldsymbol{A}]$.
- $\tilde{p}_{\mathbf{1}} = \mathbb{P}[\boldsymbol{A} = \mathbf{1}]$.

Translation: "Roughly speaking, (when $\tilde{p}$ is real stable) we show that

$$\|\mathbf{1} - \nabla\tilde{p}(\mathbf{1})\|_1 < 1 - \epsilon \quad \implies \quad \tilde{p}_{\mathbf{1}} \geq f(\epsilon, n),$$

where $f(\epsilon, n) \sim \epsilon^{2^n}$ has no dependence on m."

Metric TSP: Our Bound

Recall our bound, for $\|\mathbf{1} - \nabla p(\mathbf{1})\|_1 < 2$:

$$p_{\mathbf{1}} \geq \frac{n!}{n^n} \cdot \mathrm{Cap}_{\mathbf{1}}(p) \geq \frac{n!}{n^n} \left(1 - \frac{\|\mathbf{1} - \nabla p(\mathbf{1})\|_1}{2}\right)^n.$$

How can we apply this? Want:

$$\|\mathbf{1} - \nabla \tilde{p}(\mathbf{1})\|_1 < 1 - \epsilon \quad \implies \quad \tilde{p}_{\mathbf{1}} \geq f(\epsilon, n),$$

- Need to homogenize and transform to make degree = # variables.
- Discrepancy between $1 - \epsilon$ and 2 is reconciled in homogenization.

Implies: $\mathbb{P}[\boldsymbol{A} = \mathbf{1}] = \tilde{p}_{\mathbf{1}} \geq e^{-n}\epsilon^d$.

Problem: It could be that $d = m \implies$ dependence on m.

However: Our bounds are tight for the case of homog. deg. = # vars. Similar situation to Gurvits' original capacity bound?

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Proof Outline

A **product polynomial** is a polynomial coming from a matrix A:

$$p_A(\boldsymbol{x}) := \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j.$$

When row sums are $\mathbf{1}$ and column sums are $\boldsymbol{\alpha}$, we say $p_A \in \mathrm{Prod}_n(\boldsymbol{\alpha})$.

Theorem (Bound for product polynomials)

If $\|\mathbf{1} - \boldsymbol{\alpha}\|_1 < 2$, *then* $\displaystyle\min_{p_A \in \mathrm{Prod}_n(\boldsymbol{\alpha})} \mathrm{Cap}_{\mathbf{1}}(p_A) \geq \left(1 - \frac{\|\mathbf{1} - \boldsymbol{\alpha}\|_1}{2}\right)^n.$

Theorem (Productization)

Let p be real stable, n-variate, n-homogeneous, and set $\boldsymbol{\alpha} := \nabla p(\mathbf{1})$.
For all $\boldsymbol{x} > 0$, *there is an* $p_A \in \mathrm{Prod}_n(\boldsymbol{\alpha})$ *such that* $p_A(\boldsymbol{x}) = p(\boldsymbol{x})$.

Bound for Product Polynomials: Proof Sketch

We first need a lemma:

Lemma

For $\boldsymbol{\alpha} \in \mathbb{R}_+^n$ with $\alpha_1 + \cdots \alpha_n = n$, the following are equivalent.

1. $\|\mathbf{1} - \boldsymbol{\alpha}\|_1 < 2$.
2. $\min_{p_A \in \text{Prod}_n(\boldsymbol{\alpha})} \text{per}(A) > 0 \left(\iff \min_{p_A \in \text{Prod}_n(\boldsymbol{\alpha})} \text{Cap}_{\mathbf{1}}(p_A) > 0 \right)$.
3. $\sum_{i \in F} \alpha_i > |F| - 1$ *for all* $F \subseteq [n]$.

Proof of bound for $p_A \in \text{Prod}_n(\boldsymbol{\alpha})$:

(1) Find doubly stochastic matrix D for which $M := \frac{A - \gamma D}{1 - \gamma} \geq 0$ entrywise, and such that $\gamma \geq 0$ is maximal.
(2) Maximality of γ implies $\text{per}(M) = 0$.
(3) Use the Lemma and rearrange to obtain $\gamma \geq 1 - \frac{\|\mathbf{1} - \boldsymbol{\alpha}\|_1}{2}$.
(4) Finally, $\text{Cap}_{\mathbf{1}}(p_A) \geq \text{Cap}_{\mathbf{1}}(p_{\gamma D}) = \gamma^n$.

Productization of Real Stable Polynomials

We first need a lemma:

Lemma (Brändén)

Let p be real stable, n-variate, n-homogeneous, $p(\mathbf{1}) = 1$, and $\nabla p(\mathbf{1}) = \mathbf{1}$. Let $\lambda(\boldsymbol{x})$ denote the roots of $p(\mathbf{1}t - \boldsymbol{x})$. Then $\boldsymbol{x}$ majorizes $\lambda(\boldsymbol{x})$ for all $\boldsymbol{x}$.

Proof of productization result:

(1) First for rational $\boldsymbol{\alpha} = (\frac{k_1}{N}, \ldots, \frac{k_n}{N})$, define:

$$q(\boldsymbol{z}) := p\left(\frac{z_{1,1} + \cdots + z_{1,k_1}}{k_1}, \ldots, \frac{z_{n,1} + \cdots + z_{n,k_n}}{k_n}\right)^N.$$

(2) $q(\mathbf{1}) = 1$ and $\nabla q(\mathbf{1}) = \mathbf{1} \implies$ Lemma gives $p_D \in \mathrm{Prod}_n(\mathbf{1})$ for q.
(3) Summing blocks of the matrix D gives $p_A \in \mathrm{Prod}_n(\boldsymbol{\alpha})$ for p.
(4) For irrational $\boldsymbol{\alpha}$, use the fact that $\boldsymbol{r} \mapsto \nabla \log p(\boldsymbol{r}\boldsymbol{x})\big|_{\boldsymbol{x}=\mathbf{1}}$ maps the strict positive orthant onto the interior of $\mathrm{Newt}(p)$ and limit.

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1. Can our techniques be used to improve the metric TSP bound in general?
2. Can the productization result be extended to strongly log-concave polynomials (even log-concave)?
3. Other applications?