ONLINE LEARNING IN MDPS
PART 2

Gergely Neu
Universitat Pompeu Fabra, Barcelona

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ADVERSARIAL MDPS

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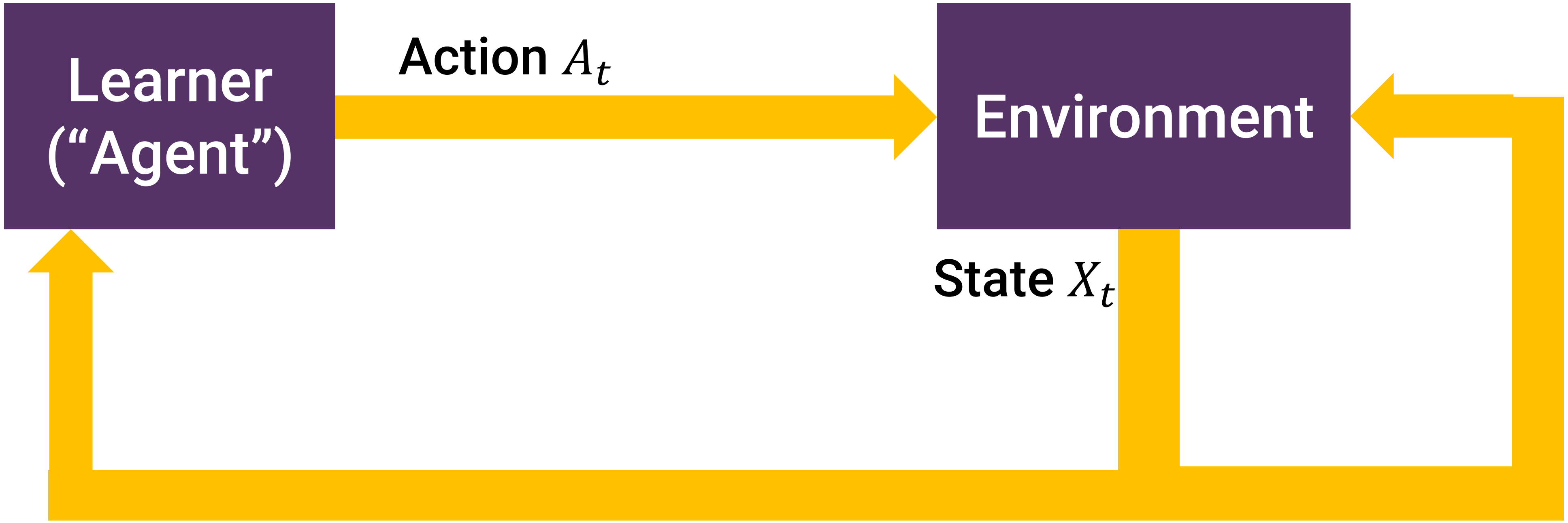
MARKOV DECISION PROCESSES



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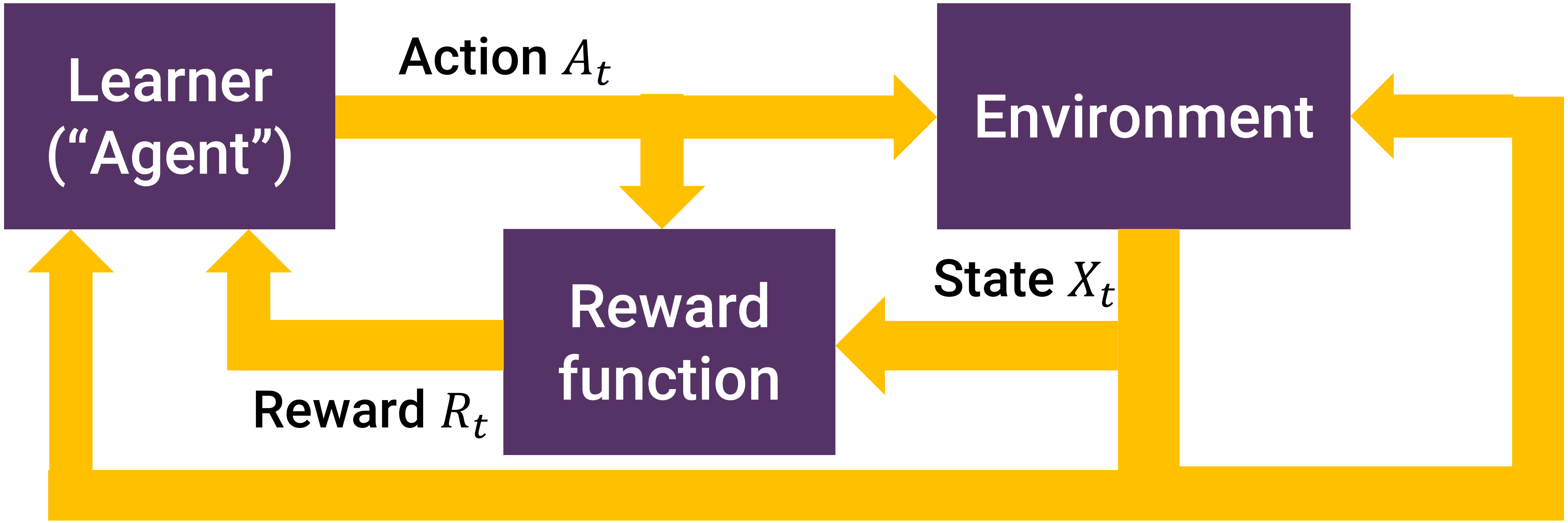

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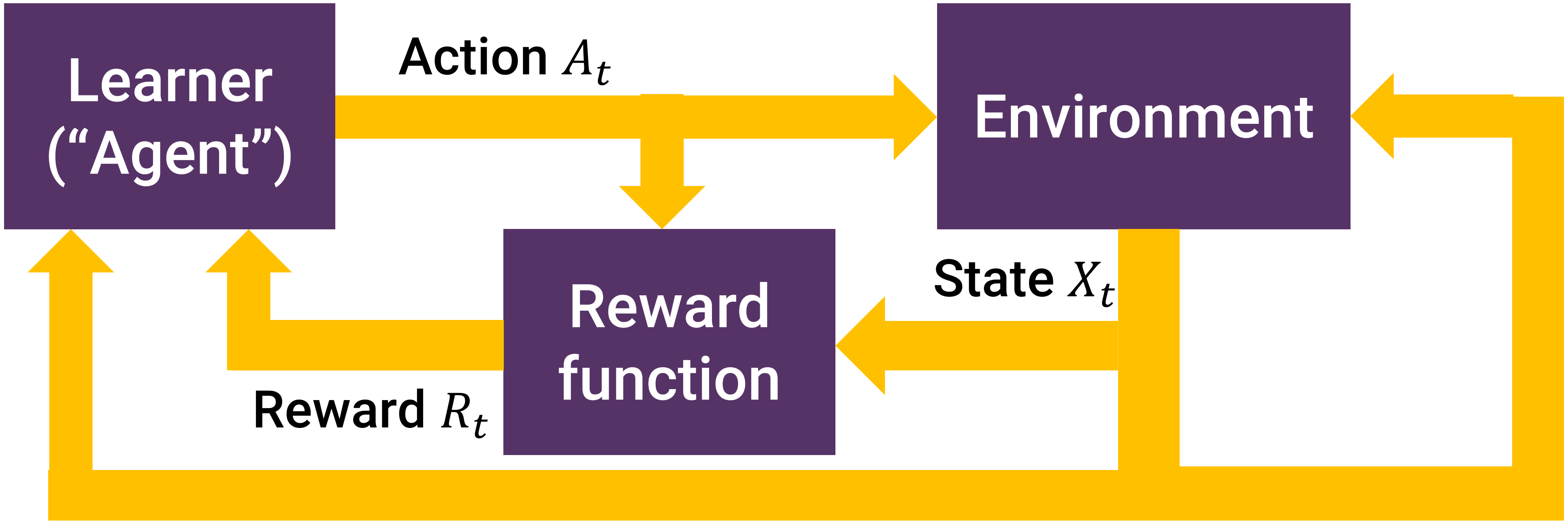
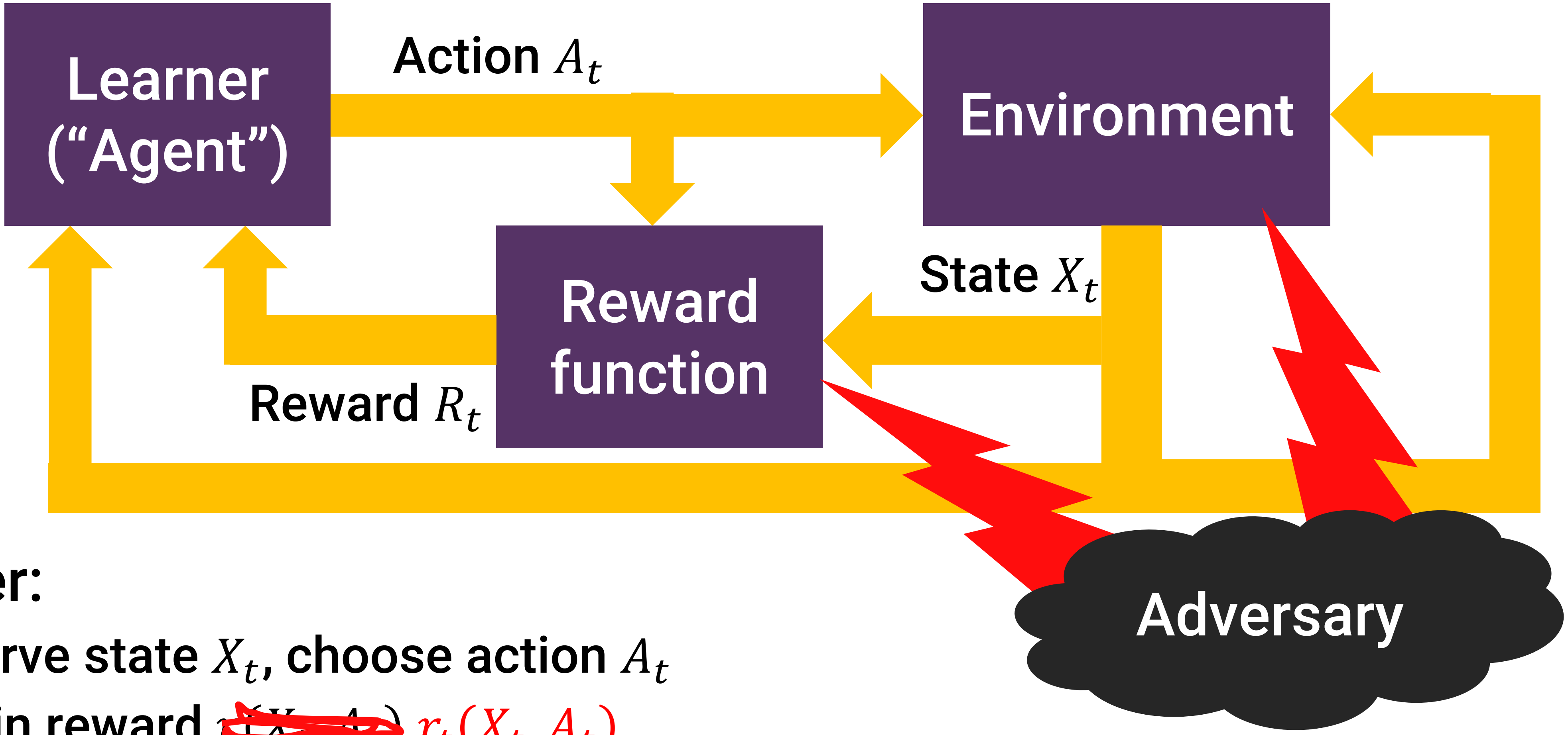


- Learner:
  - Observe state X_t, choose action A_t
  - Obtain reward $r(X_t, A_t)$
- Environment: Draw next state $X_{t+1} \sim P(\cdot\, | X_t, A_t)$

ADVERSARIAL MARKOV DECISION PROCESSES



- Learner:
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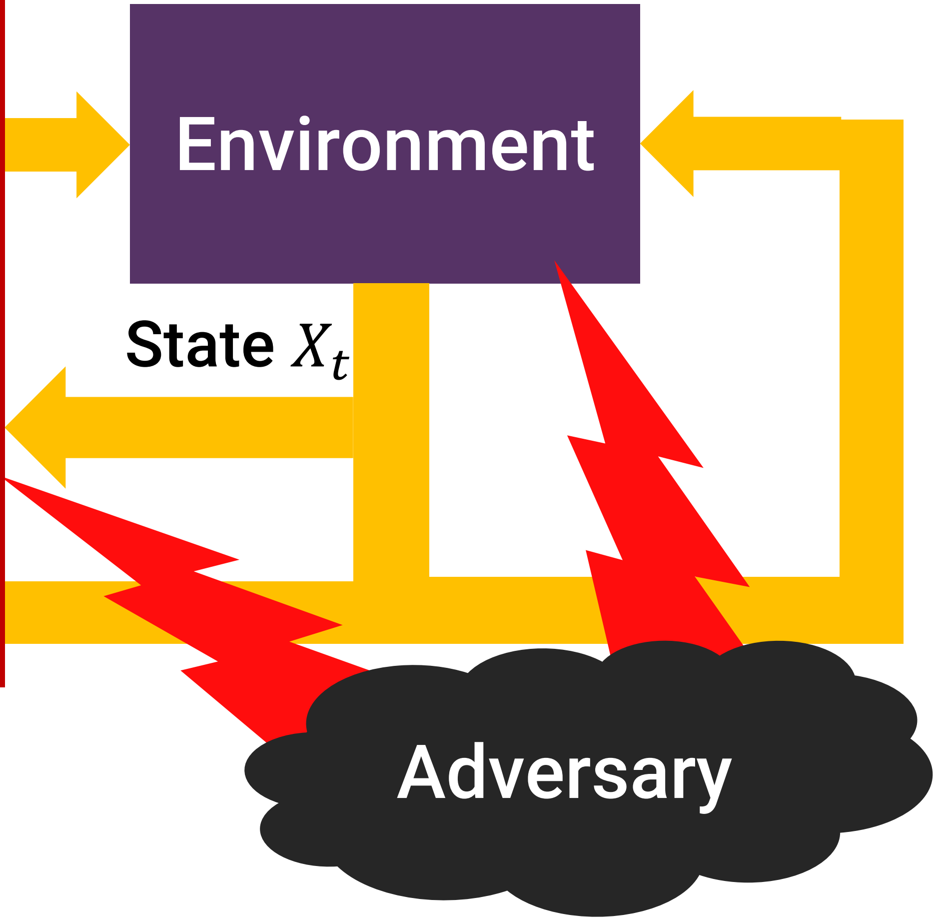
ADVERSARIAL MARKOV DECISION PROCESSES

This talk:
what is achievable when an external adversary is allowed to change the reward function and the transition function over time?



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PERFORMANCE MEASURE: REGRET

Regret

$$\mathfrak{Reg}_T(\pi) = \sum_{t=1}^{T} \mathbb{E}\Big[r_t\big(X_t^*, \pi(X_t^*)\big) - r_t(X_t, A_t)\Big],$$

where $X_1^*, X_2^*, \ldots$ is the sequence of states that would have been generated by running comparator policy π through the dynamics $P_1, P_2, \ldots$

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Goal: sublinear regret

$$\lim_{T\to\infty} \max_{\pi} \frac{\mathfrak{Reg}_T(\pi)}{T} = 0$$

OUTLINE

- Hardness results
  - Non-oblivious adversaries
  - Arbitrarily changing dynamics
- Arbitrarily changing reward functions
  - Some common ideas
  - Two algorithm families

SOME HARDNESS RESULTS

NON-OBLIVIOUS ADVERSARIES

Non-oblivious adversary:
can take history $\mathcal{H}_t = X_t, A_{t-1}, X_{t-1}, A_{t-2}, \ldots$
into account when selecting r_t and P_t

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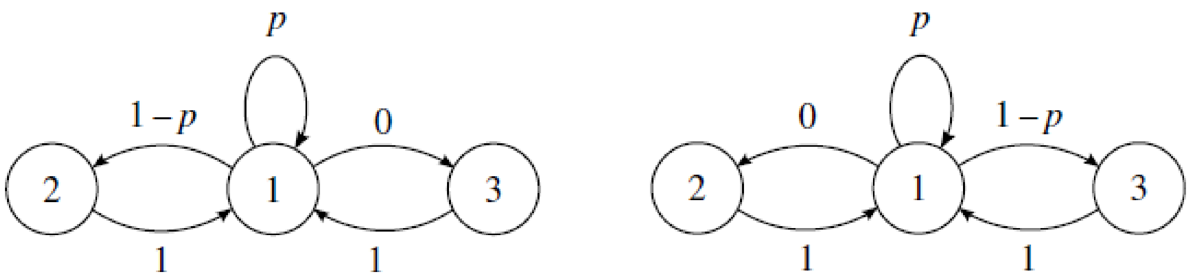
Theorem
(Yu, Mannor and Shimkin, 2009)
No algorithm can guarantee sublinear regret against a non-oblivious adversary

PROOF

Simple counterexample by Yu, Mannor and Shimkin (2009):



(a) Transition model if the agent chooses to go left.

(b) Transition model if the agent chooses to go right.

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- Reward is function of state
- $r_t(\text{default}) = 0$
- $r_t(\text{left}) = 1$ **if** $A_{t-1} = \text{right}$
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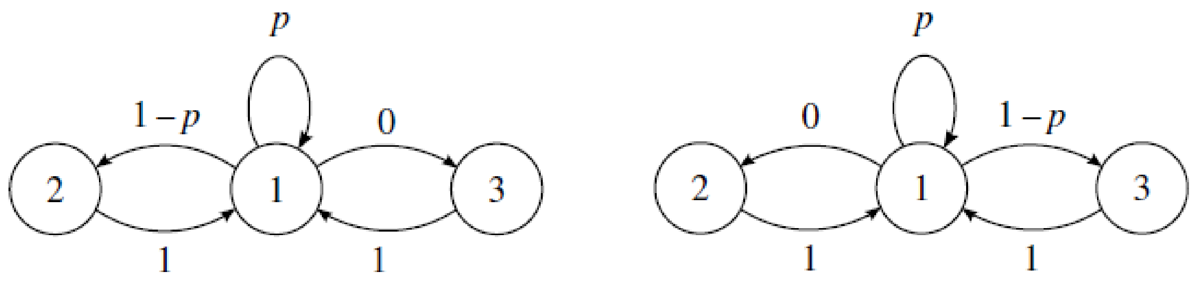

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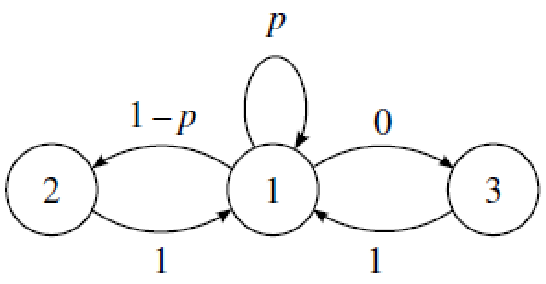

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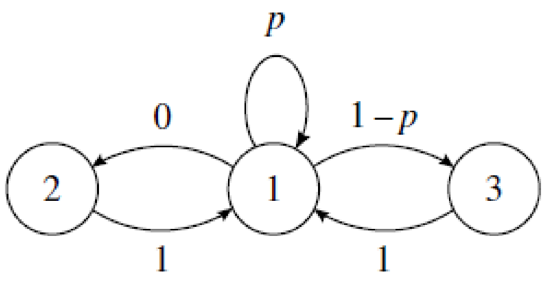

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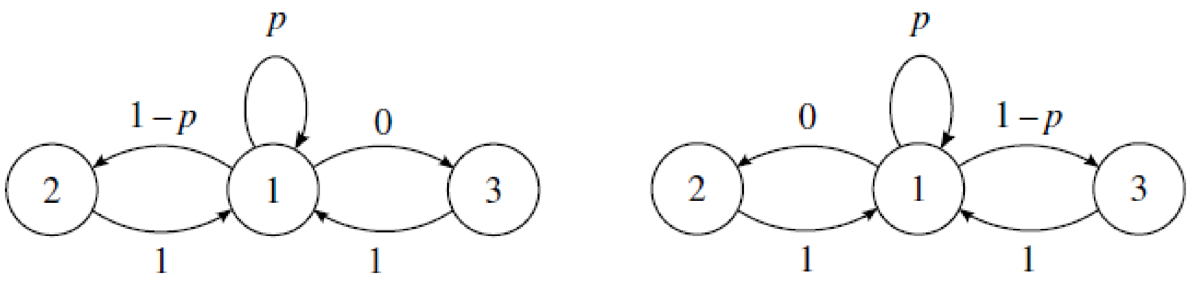

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But there is a policy π **with**

$$\mathbb{E}\left[\sum_t r_t\left(X_t^*, \pi(X_t^*)\right)\right] \geq \left(\frac{1}{2} - p\right) T$$

Either $\pi(1) = \text{left}$ or $\pi(1) = \text{right}$

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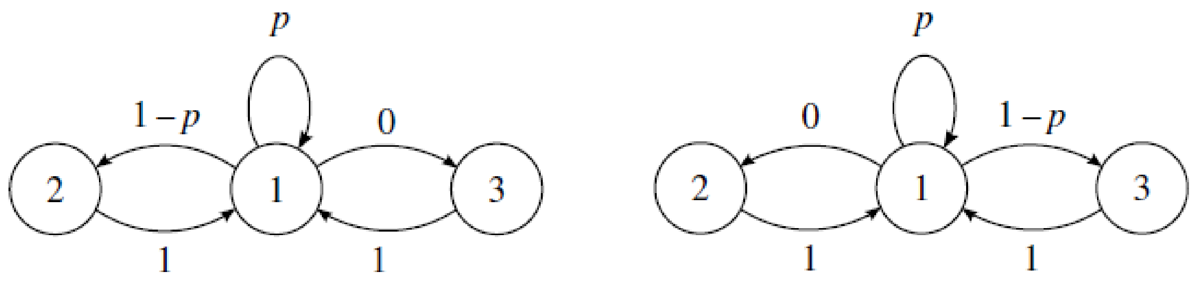

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$$\mathfrak{Reg}_T(\pi) \geq \left(\frac{1}{2} - p\right) T$$

WHAT WENT WRONG?

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Possible solutions:

- Consider "policy regret": redefine comparator to take the effect $\mathcal{H}_t \to r_t$ into account
- Consider oblivious adversaries

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OBLIVIOUS ADVERSARIES

Oblivious adversary:
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"Adversary $\approx$ nature":
it can (mis)behave arbitrarily, but doesn't care about what you do

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“Adversary $\approx$ nature”:
it can (mis)behave arbitrarily, but doesn’t care about what you do

Can we guarantee sublinear regret now?

LEARNING WITH CHANGING TRANSITIONS IS HARD

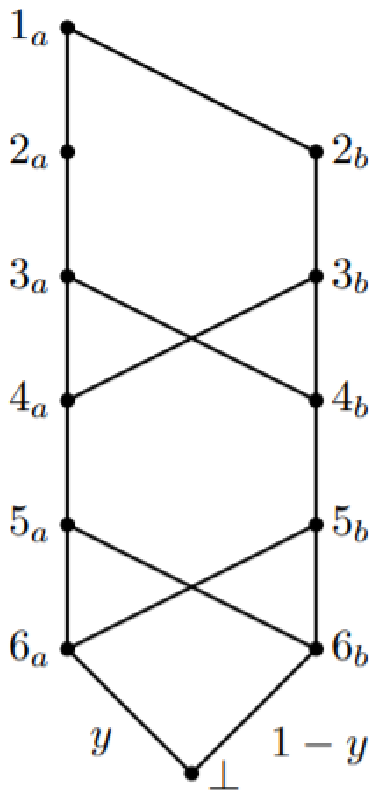
Learning against an oblivious adversary can still be computationally hard when the transition function is allowed to change!

Theorem

(Abbasi-Yadkori et al., 2013)
There is an adversarial MDP where achieving sublinear regret is computationally hard.

PROOF CONSTRUCTION

- **Idea:** learning of noisy parities can be formulated as an MDP with changing transition functions & rewards!
- $O(\text{poly}(n)T^{1-\alpha})$ regret $\Rightarrow O\left(\frac{poly(n)}{\varepsilon^{1/\alpha}}\right)$ excess risk, conjectured to be computationally hard to achieve
- Construction: an instance $x \in \{0,1\}^n$ corresponds to a deterministic transition graph with rewards determined by the label y



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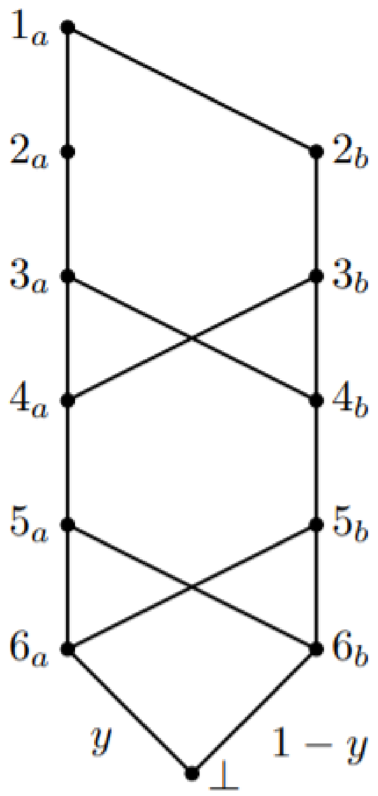

Corresponds to an oblivious adversary that picks (P_t, r_t) jointly!

SLOWLY CHANGING MDPS

Very recent work by Gajane et al. (2019), Cheung et al. (2020):

- define reward and transition variation as

$$V_T^r = \sum_{t=1}^{T} \max_{x,a} |r_t(x,a) - r_{t+1}(x,a)|$$

$$V_T^P = \sum_{t=1}^{T} \max_{x,a} \|P_t(\cdot\,|x,a) - P_{t+1}(\cdot\,|x,a)\|_1$$

- regret bounds of $O\left((V_T^P + V_T^r)^{1/3} T^{2/3}\right)$ are possible
- algorithm: UCRL + forgetting old data

ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

WHERE IT ALL STARTED...

Experts in a Markov Decision Process

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Online Markov Decision Processes

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Math of OR 2009

FORMAL PROTOCOL

Online learning in a fixed MDP

For each round $t = 1,2,\ldots,T$

- Learner observes state $X_t \in \mathcal{X}$
- Learner takes action $A_t \in \mathcal{A}$
- Adversary selects reward function $r_t\colon \mathcal{X} \times \mathcal{A} \to [0,1]$
- Learner earns reward $R_t = r_t(X_t, A_t)$
- Learner observes feedback
  - Full information: r_t
  - Bandit feedback: R_t
- Environment produces new state $X_{t+1} \sim P(\cdot\,|X_t, A_t)$

FORMAL PROTOCOL

Online learning in a fixed MDP

For each round $t = 1,2, \dots, T$

- Learner observes state $X_t \in \mathcal{X}$
- Learner selects stochastic policy π_t
- Learner takes action $A_t \sim \pi_t(\cdot|X_t)$
- Adversary selects reward function $r_t: \mathcal{X} \times \mathcal{A} \to [0,1]$
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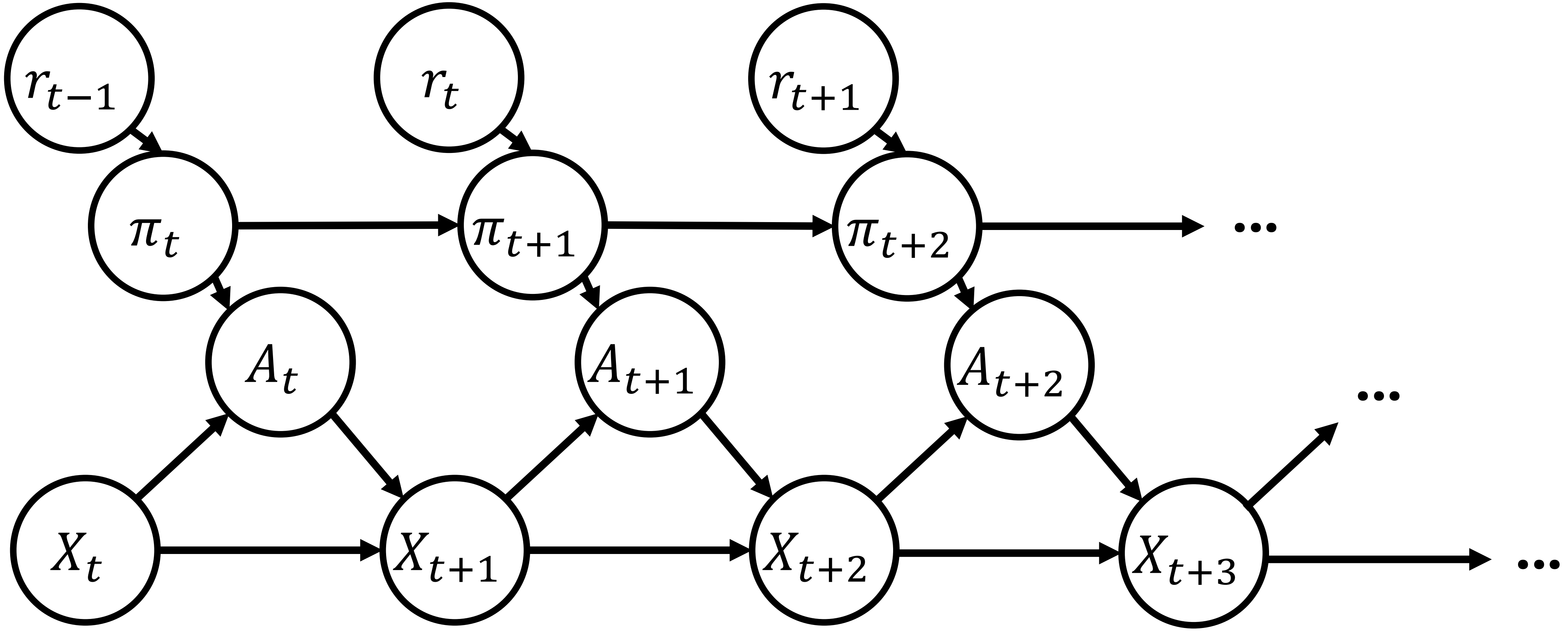
Stochastic policy: $\pi(a|x) = \mathbb{P}[A_t = a|X_t = x]$

TEMPORAL DEPENDENCES

Main challenge: dependence between consecutive time steps



NB this graph is accurate for full information feedback; bandit is a bit more complicated

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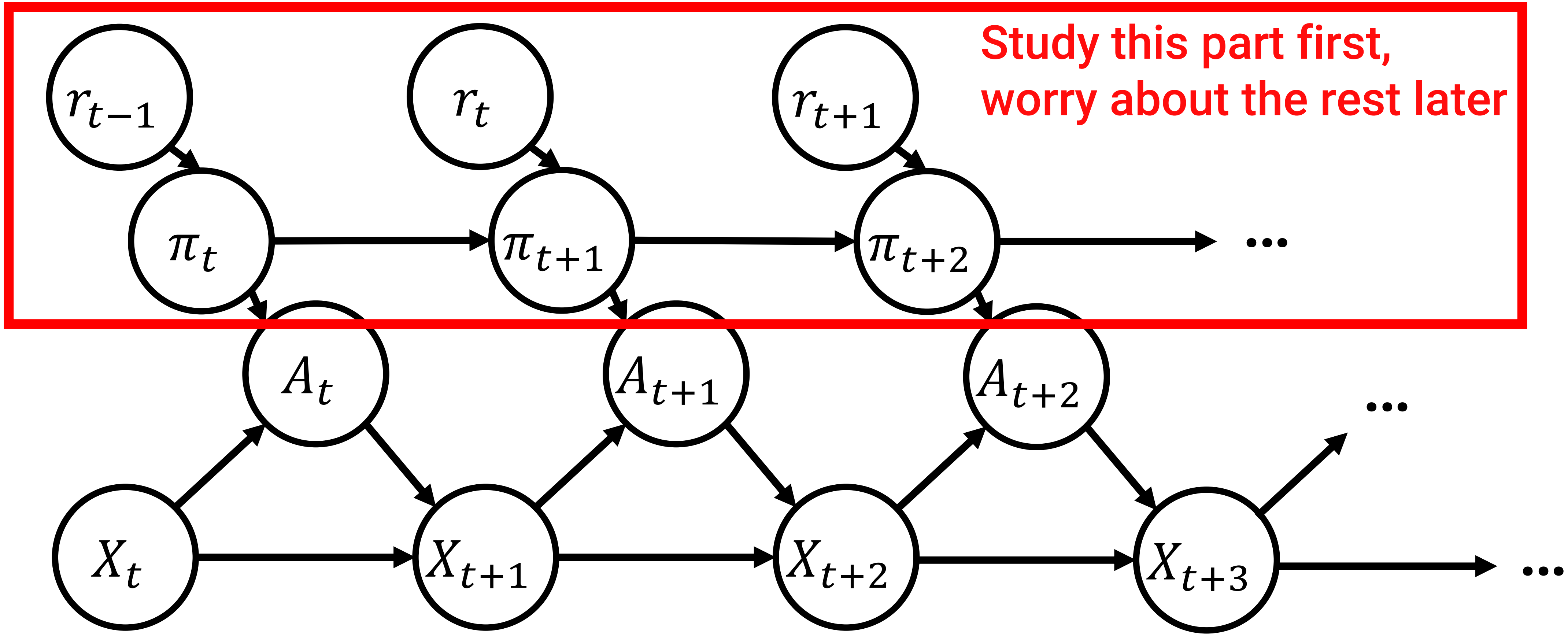

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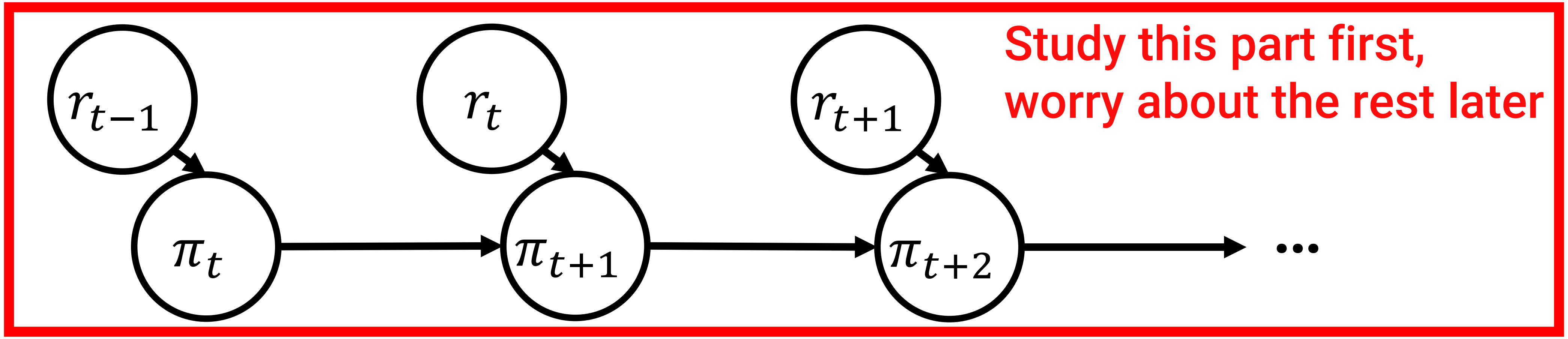

"Pretend that every policy reaches its stationary distribution immediately!"

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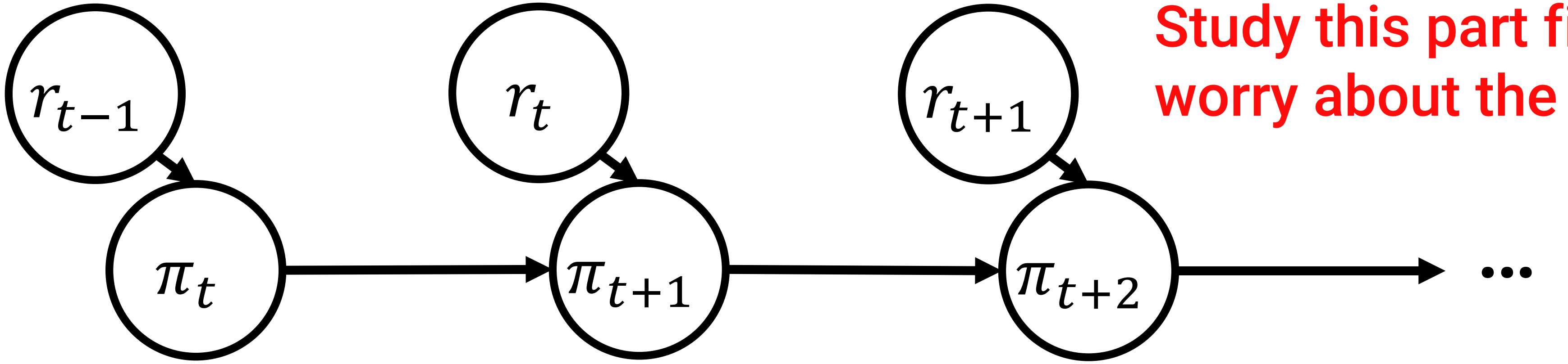

Study this part first,
worry about the rest later

"Pretend that every policy reaches its **stationary distribution** immediately!"

Def: stationary distribution of policy π:

$$\mu_\pi(x,a) = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}_{\{X_k = x, A_k = a\}}$$

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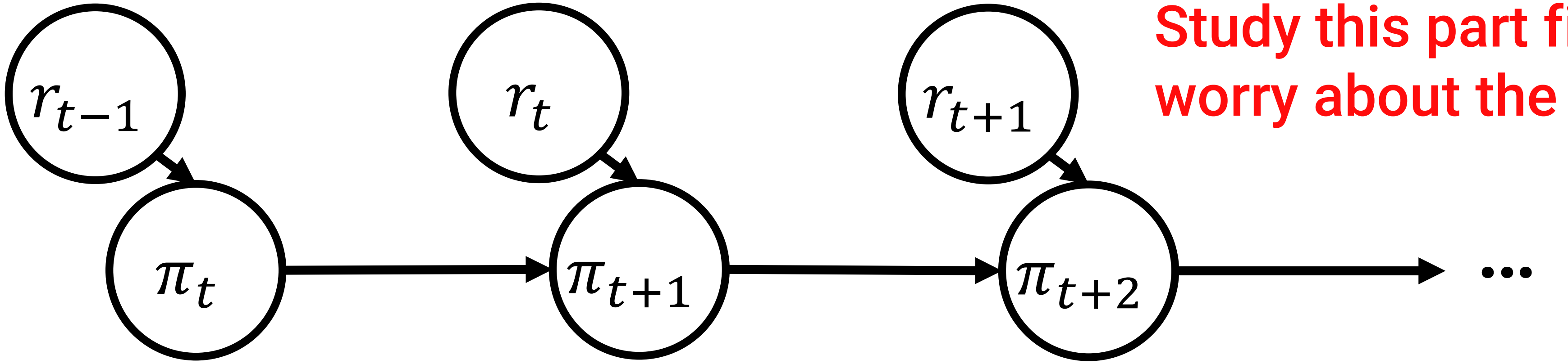

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Assumption: 1-step mixing $\forall \pi$

$$\|(\nu - \nu')P_\pi\|_1 \le e^{1/\tau} \|\nu - \nu'\|_1$$

REGRET DECOMPOSITION

- Define
  $\nu_t(x,a) = \mathbb{P}[X_t = x, A_t = a]$ and $\nu_t^*(x,a) = \mathbb{P}[X_t^* = x, A_t^* = a]$
  $\mu_t = \mu_{\pi_t}$, stationary distribution induced by policy π_t
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$$= \sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle + \sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle + \sum_{t=1}^{T} \langle \mu_t - \nu_t, r_t \rangle$$

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$$= \sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle + \underbrace{\sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle}_{\text{“stationarized regret”}} + \sum_{t=1}^{T} \langle \mu_t - \nu_t, r_t \rangle$$

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$$= \underbrace{\sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle}_{\text{“comparator drift”}} + \underbrace{\sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle}_{\text{“stationarized regret”}} + \underbrace{\sum_{t=1}^{T} \langle \mu_t - \nu_t, r_t \rangle}_{\text{“learner drift”}}$$

THE DRIFT TERMS

- For the comparator, fast mixing is guaranteed by assumption:

$$\sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle \leq \sum_{t=1}^{T} \|\nu_t^* - \mu^*\|_1 \leq \sum_{t=1}^{T} e^{-t/\tau} \|\nu_1^* - \mu^*\|_1 \leq 2\tau + 2$$

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- The other term is small if the policies change slowly:

Lemma

If $\max_x \|\pi_t(\cdot\,|x) - \pi_{t-1}(\cdot\,|x)\|_1 \leq \varepsilon$ for all t, then

$$\sum_{t=1}^{T} \|\mu_t - \nu_t\|_1 \leq (\tau + 1)^2 \varepsilon T + 2e^{-T/\tau}$$

"ν_t tracks μ_t if policies change slowly"

ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

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Local-to-global regret decomposition	Reduction to online linear optimization

ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

Local-to-global regret decomposition

Reduction to online linear optimization

LOCAL-TO-GLOBAL REGRET DECOMPOSITION

- Idea by Even-Dar, Kakade and Mansour (2005,2009) based on the performance difference lemma:

Lemma

Let π, π' be two arbitrary policies, r a reward function and Q^π and V^π be the value functions corresponding to r and π. Then,

$$\langle \mu_{\pi'} - \mu_\pi, r \rangle = \langle \mu_{\pi'}, Q^\pi - V^\pi \rangle$$

LOCAL-TO-GLOBAL REGRET DECOMPOSITION

Apply with $r = r_t, \pi = \pi_t$ and $\pi' = \pi^*$:

$$\langle \mu^* - \mu_t, r \rangle = \sum_x \mu^*(x) \sum_a \big(\pi^*(a|x) - \pi_t(a|x)\big) Q_t(x,a)$$

LOCAL-TO-GLOBAL REGRET DECOMPOSITION

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Q-function of π_t with reward function r_t

LOCAL-TO-GLOBAL REGRET DECOMPOSITION

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Local regret in state x with reward function $Q_t(x,\cdot)$

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Algorithm idea:
run a local regret-minimization algorithm in each state x with reward function $Q_t(x,\cdot)$!

THE MDP-EXPERT ALGORITHM

MDP-E

For each round $t = 1, 2, \dots, T$

- Observe state X_t
- Take action $A_t \sim \pi_t(\cdot\,|X_t)$
- Observe reward function r_t
- Calculate value functions as solution to $Q_t(x,a) = r_t - \langle \mu_t, r_t \rangle + \sum_{x'} P(x'|x,a) V_t(x')$
- For all x, feed $Q_t(x,\cdot)$ to expert algorithm $\mathfrak{Alg}(x)$

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- For all x, feed $Q_t(x,\cdot)$ to expert algorithm $\mathfrak{Alg}(x)$
- **Example:** $\mathfrak{Alg}$ = Exponential weights

$$\pi_{t+1}(a|x) \propto \pi_t(a|x) \cdot e^{\eta Q_t(x,a)}$$

GUARANTEES FOR MDP-E

Theorem

(Even-Dar et al., 2009, Neu et al., 2014)
If $\mathfrak{Alg}(x)$ guarantees a regret bound of B_T for rewards bounded in $[0,1]$, the stationarized regret of MDP-E satisfies

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r \rangle \leq \tau B_T$$

Proof is obvious given the regret decomposition.

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Theorem

If $\mathfrak{Alg}(x)$=EWA, the regret of MDP-E satisfies

$$\mathfrak{Reg}_T = O\left(\sqrt{\tau^3 T \log|\mathcal{A}|}\right)$$

Proof is obvious given the regret decomposition.

BANDIT FEEDBACK

Addressed in Neu, György, Szepesvári and Antos (2010,2014): replace r_t by an unbiased estimator

$$\hat{r}_t(x,a) = \frac{r_t(x,a)}{\mu_t^N(x,a)} \mathbb{I}\{(X_t, A_t) = (x,a)\},$$

with $\mu_t^N(x,a) = \mathbb{P}[(X_t, A_t) = (x,a) | \mathcal{H}_{t-N}]$

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Remember Exp3?

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Theorem

If $\mathfrak{Alg}(x)$=EWA, the regret of MDP-Exp3 satisfies

$$\mathfrak{Reg}_T = O\left(\sqrt{\tau^3 T |\mathcal{A}| \log|\mathcal{A}| / \beta}\right)$$

Assumption: $\mu_\pi(x) \geq \beta$ for all π, x

ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

Local-to-global regret decomposition	Reduction to online linear optimization

ONLINE LINEAR OPTIMIZATION

Notice: stationarized regret = regret in an OLO problem!

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle$$

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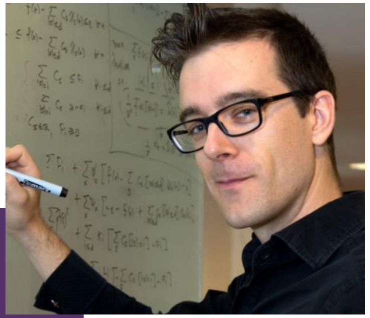

Algorithm idea:

run an OLO algorithm with the set of all stationary distributions as decision set!

$$\mathcal{U} = \left\{ \mu \in \Delta_{\mathcal{X}\times\mathcal{A}} : \sum_{a} \mu(x,a) = \sum_{x',a'} P(x|x',a')\mu(x',a') \right\}$$

ONLINE MIRROR DESCENT

- In each round, update stationary distribution

$$\mu_{t+1} = \arg\max_{\mu \in \mathcal{U}} \left\{ \langle \mu, r_t \rangle - \frac{1}{\eta} D(\mu | \mu_t) \right\}$$

and extract policy $\pi_{t+1}(a|x) \propto \mu_{t+1}(x, a)$

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and extract policy $\pi_{t+1}(a|x) \propto \mu_{t+1}(x,a)$
- Choosing the regularizer:
  - Relative entropy: $D(\mu|\nu) = \sum_{x,a} \mu(x,a) \log \frac{\mu(x,a)}{\nu(x,a)}$
    ⇒ "Online Relative Entropy Policy Search" (Zimin and Neu, 2013, Dick, György and Szepesvári, 2014)
  - Conditional relative entropy: $D(\mu|\nu) = \sum_{x,a} \mu(x,a) \log \frac{\pi_\mu(a|x)}{\pi_\nu(a|x)}$
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THE ONLINE REPS ALGORITHM

O-REPS

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  $$\min_V \log \sum_{x,a} \mu_t(x,a) e^{\eta\left(r_t(x,a)+\sum_{x'} P(x'|x,a)V(x')-V(x)\right)}$$
- Update stationary distribution as
  $$\mu_{t+1}(x,a) = \mu_t(x,a) e^{\eta\left(r_t(x,a)+\sum_{x'} P(x'|x,a)V(x')-V(x)\right)}$$

Algorithm inspired by Peters, Mülling and Altün (2010)

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Unconstrained convex minimization

- Update stationary distribution as

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Algorithm inspired by Peters, Mülling and Altün (2010)

GUARANTEES FOR O-REPS

Theorem

(Zimin and Neu, 2013, Dick et al. 2014)
The stationarized regret of O-REPS satisfies

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r \rangle \leq \sqrt{T \log|\mathcal{X}||\mathcal{A}|}$$

Theorem

The regret of O-REPS satisfies

$$\mathfrak{Reg}_T = O\left(\sqrt{\tau T \log|\mathcal{X}||\mathcal{A}|}\right)$$

Proof is based on standard OLO analysis.

BANDIT FEEDBACK

Addressed in Zimin and Neu (2013) in **episodic MDPs**:
replace r_t by an unbiased estimator

$$\hat{r}_t(x,a) = \frac{r_t(x,a)}{q_t(x,a)} \mathbb{I}\{(x,a) \text{ visited in episode } t\},$$

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If $\mathfrak{Alg}(x)$=EWA, the regret of MDP-Exp3 satisfies

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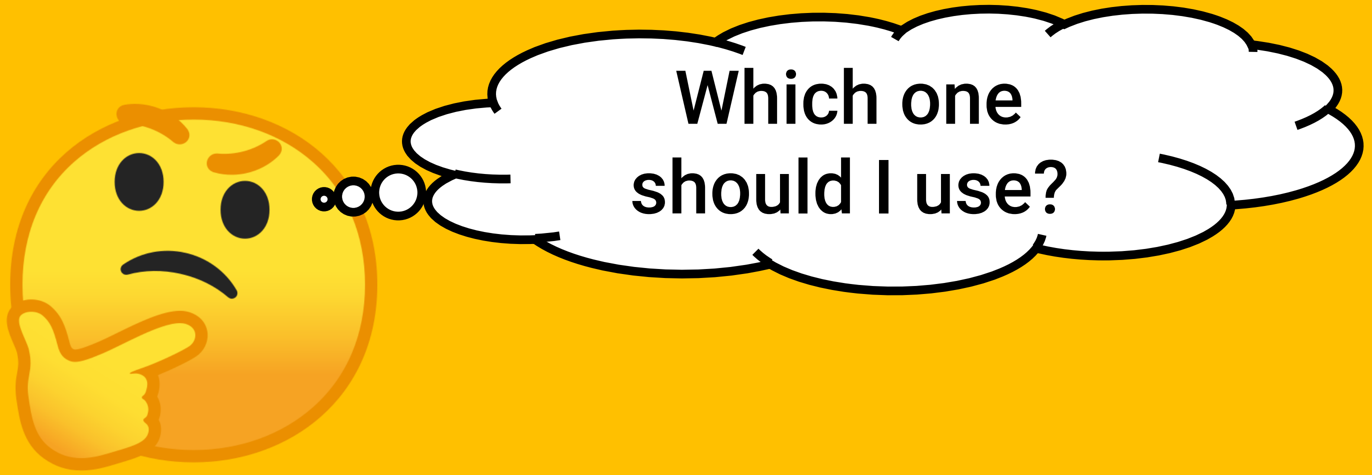
Local-to-global regret decomposition	Reduction to online linear optimization



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COMPARISON OF GUARANTEES

	MDP-E	O-REPS
Full information	$\sqrt{\tau^3 T \log\lvert\mathcal{A}\rvert}$	$\sqrt{\tau T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$
Bandit feedback	$\sqrt{\tau^3 \lvert\mathcal{A}\rvert T \log\lvert\mathcal{A}\rvert / \beta}$	???
Full information (episodic case)	$H^2\sqrt{T \log\lvert\mathcal{A}\rvert}$	$H\sqrt{T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$
Bandit feedback (episodic case)	$H^2\sqrt{\lvert\mathcal{A}\rvert T \log\lvert\mathcal{A}\rvert / \beta}$	$\sqrt{H\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$

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Bandit feedback (episodic case)	$H^2\sqrt{\lvert\mathcal{A}\rvert T \log\lvert\mathcal{A}\rvert / \beta}$	$\sqrt{H\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$

\+ MDP-E works well with function approximation for Q-function

\+ O-REPS can easily handle model constraints and extensions

MDP-E WITH FUNCTION APPROXIMATION

MDP-E only needs a good approximation of the action-value function $\widehat{Q}_t \approx Q^{\pi_t}$ to define its policy

$$\pi_{t+1}(a|x) \propto \exp\left(\eta \sum_{k=1}^{t} \widehat{Q}_k(x,a)\right)$$

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- POLITEX (Abbasi-Yadkori et al., 2019):
  use LSPE to estimate Q^{π_t} with linear FA
  regret = $O(T^{3/4} + \varepsilon_0 T)$
- OPPO (Cai et al., 2019)
  use LSPE to estimate Q^{π_t} with **realizable** linear FA
  regret = $O(\sqrt{T})$

MDP-E WITH FUNCTION APPROXIMATION

MDP-E only needs a good approximation of the action-value function $\widehat{Q}_t \approx Q^{\pi_t}$ to define its policy

$$\pi_{t+1}(a|x) \propto \exp\left(\eta \sum_{k=1}^{t} \widehat{Q}_k(x,a)\right)$$

+ MDP-E is essentially identical to the "Trust-Region Policy Optimization" (TRPO) algorithm of Schulman et al. (2015), as shown by Neu, Jonsson and Gómez (2017)!!!

O-REPS WITH UNCERTAIN MODELS

O-REPS can easily accommodate uncertainties in the transition model by extending the decision set:

$$\mathcal{U} = \left\{ \mu \in \Delta_{\mathcal{X}\times\mathcal{A}} : \sum_{a} \mu(x,a) = \sum_{x',a'} P(x|x',a')\mu(x',a') \right\}$$

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Confidence set of transition models

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Confidence set of transition models

UC-O-REPS by Rosenberg and Mansour (2019)

Extended to bandit feedback by Jin et al. (2020):

$$\hat{r}_t(x,a) = \frac{r_t(x,a)}{u_t(x,a)} \mathbb{I}\{(x,a) \text{ visited in episode } t\},$$

with $u_t(x,a) > q_t(x,a) = \mathbb{P}[(x,a) \text{ visited in episode } t | \mathcal{H}_{t-1}]$ w.h.p.

OUTLOOK

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- Open problems:
  - Lower bounds? Right scaling with τ? Is uniform mixing necessary?
  - Large state spaces and function approximation?
  - Practical algorithms?

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Relevance to practice of RL?

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  - Lower bounds? Right scaling with τ? Is uniform mixing necessary?
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  - Practical algorithms?

Relevance to practice of RL?

- Online learning algorithms are robust! Main tool: regularization
- Better understanding of regularization tools $\Rightarrow$ better algorithms!
- Remember: TRPO = MDP-E!


W95
Online Markov Decision Processes under Bandit Feedback
R L
A I
Abstract
Assumptions
Main result
Definitions
Proof
Online Markov Decision Processes
The learning problem
pf

Thanks!!!

REFERENCES

- Yu, J. Y., Mannor, S., & Shimkin, N. (2009). Markov decision processes with arbitrary reward processes. *Mathematics of Operations Research*, *34*(3), 737-757.
- Abbasi-Yadkori, Y., Bartlett, P. L., Kanade, V., Seldin, Y., & Szepesvári, Cs. (2013). Online learning in Markov decision processes with adversarially chosen transition probability distributions. In *Advances in neural information processing systems* (pp. 2508-2516).
- Gajane, P., Ortner, R., & Auer, P. (2019). Variational Regret Bounds for Reinforcement Learning. In *Uncertainty in Artificial Intelligence.*
- Cheung, W. C., Simchi-Levi, D., & Zhu, R. (2020). Reinforcement Learning for Non-Stationary Markov Decision Processes: The Blessing of (More) Optimism. In *International Conference on Machine Learning*.
- Even-Dar, E., Kakade, S. M., & Mansour, Y. (2005). Experts in a Markov decision process. In *Advances in neural information processing systems* (pp. 401-408).
- Even-Dar, E., Kakade, S. M., & Mansour, Y. (2009). Online Markov decision processes. *Mathematics of Operations Research*, *34*(3), 726-736.

REFERENCES

- Neu, G., Antos, A., György, A., & Szepesvári, C. (2010). Online Markov decision processes under bandit feedback. In *Advances in Neural Information Processing Systems* (pp. 1804-1812).
- Peters, J., Mülling, K., & Altun, Y. (2010). Relative entropy policy search. In *AAAI* (Vol. 10, pp. 1607-1612).
- Zimin, A., & Neu, G. (2013). Online learning in episodic Markovian decision processes by relative entropy policy search. In *Advances in neural information processing systems* (pp. 1583-1591).
- Dick, T., György, A., & Szepesvári, Cs. (2014). Online Learning in Markov Decision Processes with Changing Cost Sequences. In *International Conference on Machine Learning* (pp. 512-520).
- Abbasi-Yadkori, Y., Bartlett, P., Bhatia, K., Lazic, N., Szepesvári, Cs., & Weisz, G. (2019). POLITEX: Regret bounds for policy iteration using expert prediction. In *International Conference on Machine Learning* (pp. 3692-3702).
- Cai, Q., Yang, Z., Jin, C., & Wang, Z. (2019). Provably efficient exploration in policy optimization. *arXiv preprint arXiv:1912.05830.*

REFERENCES

- Neu, G., Jonsson, A., & Gómez, V. (2017). A unified view of entropy-regularized Markov decision processes. *arXiv preprint arXiv:1705.07798*.
- Rosenberg, A., & Mansour, Y. (2019, May). Online Convex Optimization in Adversarial Markov Decision Processes. In *International Conference on Machine Learning* (pp. 5478-5486).
- Rosenberg, A., & Mansour, Y. (2019). Online stochastic shortest path with bandit feedback and unknown transition function. In *Advances in Neural Information Processing Systems* (pp. 2212-2221).
- Jin, C., Jin, T., Luo, H., Sra, S., & Yu, T. (2020). Learning adversarial Markov decision processes with bandit feedback and unknown transition. In *International Conference on Machine Learning* (pp. 1369-1378).