Phase transitions in the complexity of simulating random shallow quantum circuits

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Simons workshop

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Complexity from entanglement

Original motivation for quantum computing [Feynman '82]

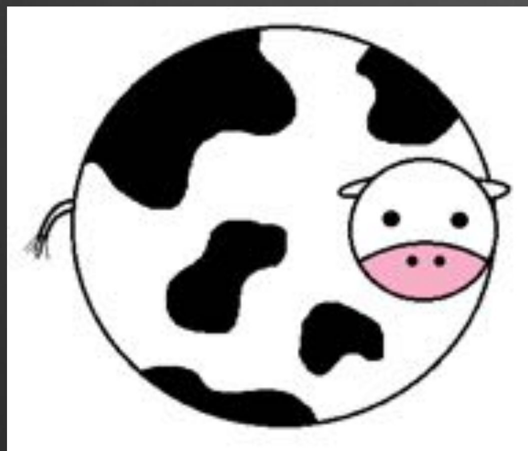
Nature isn't classical, dammit, and if you want to make a simulation of Nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy.

N systems in product state → O(N) degrees of freedom
N entangled systems → exp(N) degrees of freedom

Describes cost of simulating dynamics or even describing a state.

This talk: do typical quantum dynamics achieve this?

easier quantum simulations



1. solve trivial special case (e.g. non-interacting theory)
2. treat corrections to theory as perturbations

easier quantum simulation

Lightly entangling dynamics

product states + non-interacting gates are easy.
Cost grows exponentially with # of entangling gates.

Stabilizer circuits

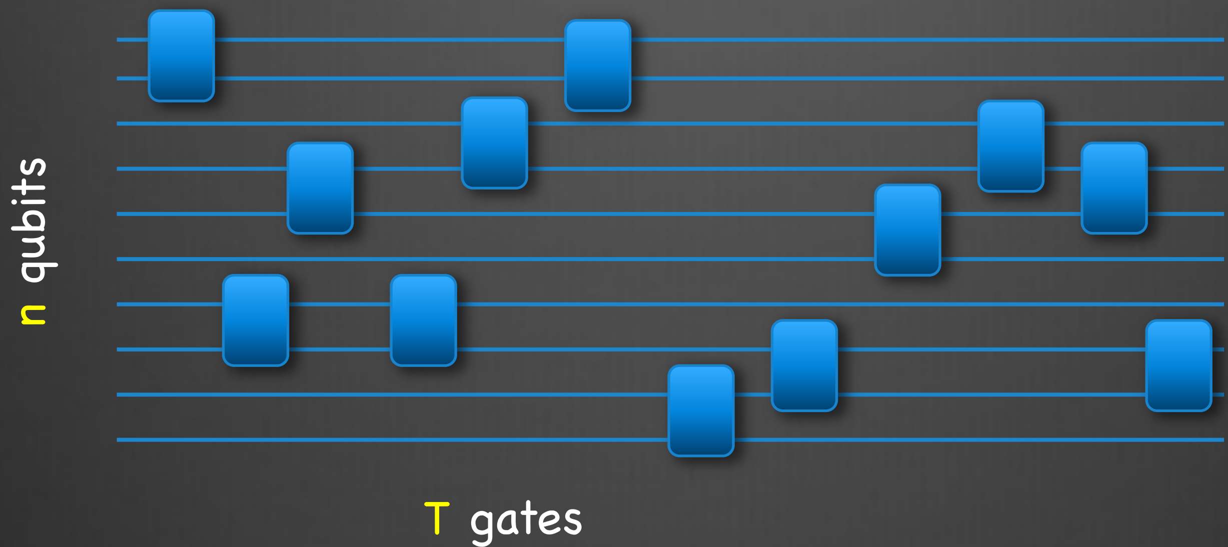
Poly-time simulation of stabilizer circuits,
growing exponentially with # of non-stabilizer gates.

Likewise for matchgates / non-interacting fermions.

Ground states of 1-D systems

Effort grows exponentially with correlation length.

quantum circuits



Classical simulation possible in time O(T)·exp(k), where

- k = treewidth [Markov-Shi '05]
- k = max # of gates crossing any single qubit [Yoran-Short '06, Jozsa '06]

+ Complexity interpolates between linear and exponential.

− Treating all gates as "potentially entangling" is too pessimistic.

noisy dynamics?

Time evolution of quantum systems

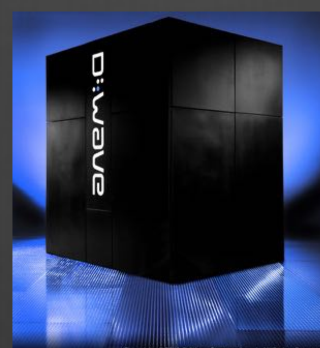
$$\frac{d\rho}{dt} = -i(H\rho - \rho H) + \text{noise terms that are linear in } \rho$$

noise per gate

0 — ideal QC

10^{-2}-ish — FTQC possible

?



?

≈0.3

classical simulation possible — 1

conjectured to exhibit phase transition
(possibly with intermediate phases)

phase transitions?

Complexity smoothly increases with
-entanglement
-correlation length
-# of non-stabilizer gates

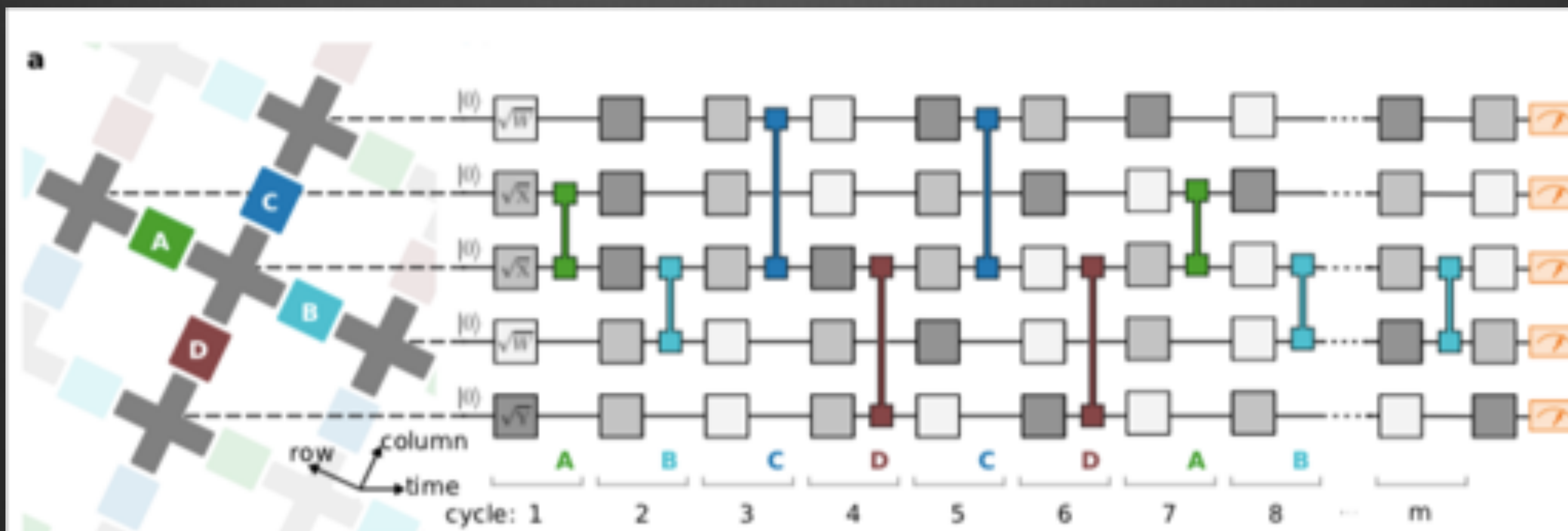
Complexity jumps discontinuously with
-noise rate

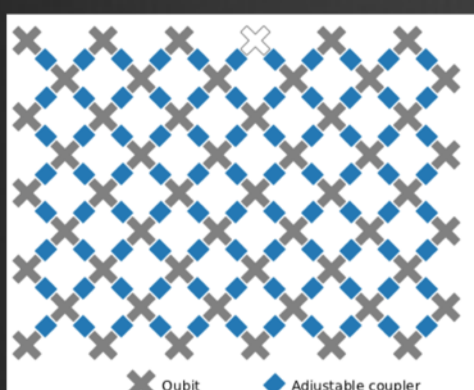
Today: what about circuit depth?

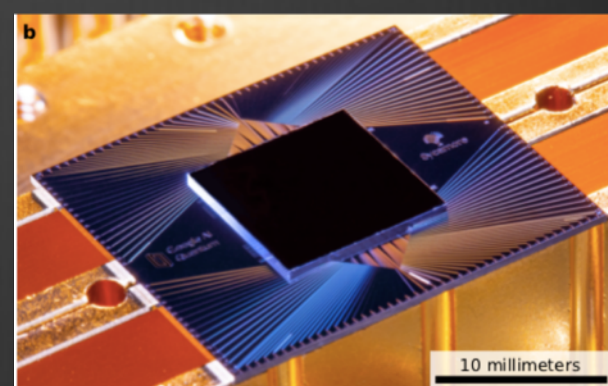
Quantum supremacy using a programmable superconducting processor

Google AI Quantum and collaborators†

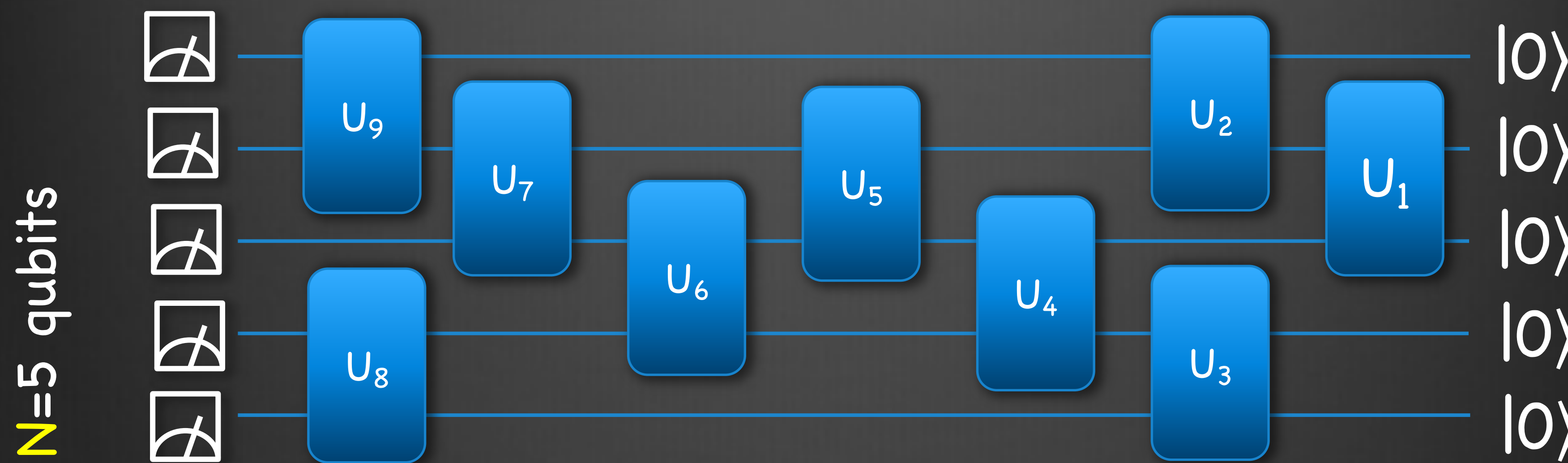
task chosen to favor quantum computers and clear comparison







quantum circuits



depth T=7

$$p(z) = p(z_1, z_2, z_3, z_4, z_5) = |\langle z_1 z_2 z_3 z_4 z_5 | \, U_9 U_8 U_7 U_6 U_5 U_4 U_3 U_2 U_1 \, |00000\rangle|^2$$

Other parameters: connectivity, # of gates, fidelity.

random circuit sampling

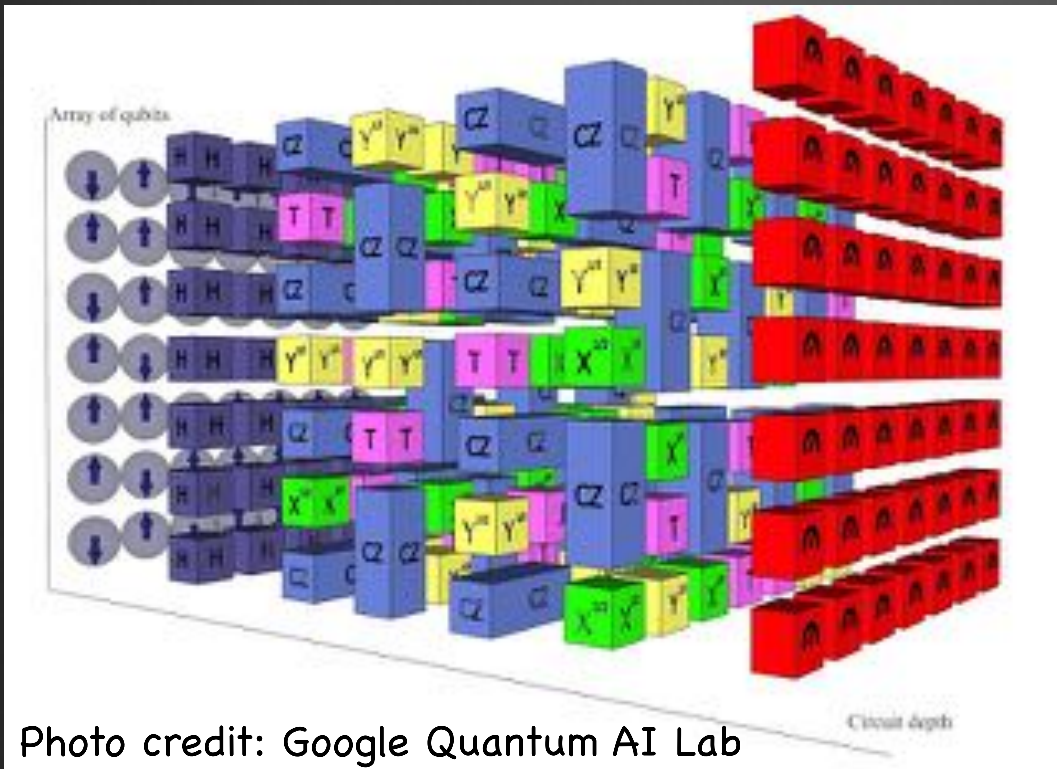


Photo credit: Google Quantum AI Lab

<u>Conjecture:</u>
Output distribution p(z) is hard to sample from on classical computer.

Google used N=53 qubits in 2D geometry with T=20.

Conjecture: $T \geq \sqrt{N}$ → classical simulation time exp(N).
[Aaronson, Bremner, Jozsa, Montanaro, Shepherd, ...]

low-depth circuits

Google proposal is $\sqrt{N} \times \sqrt{N}$ grid for depth $T \sim \sqrt{N}$.

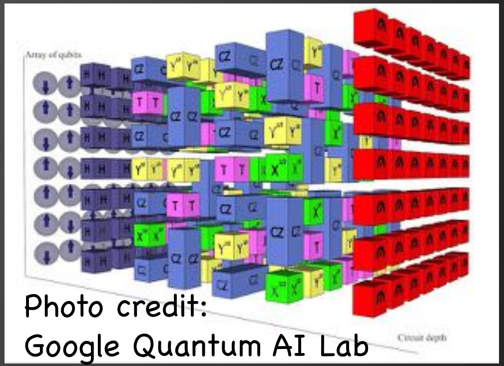
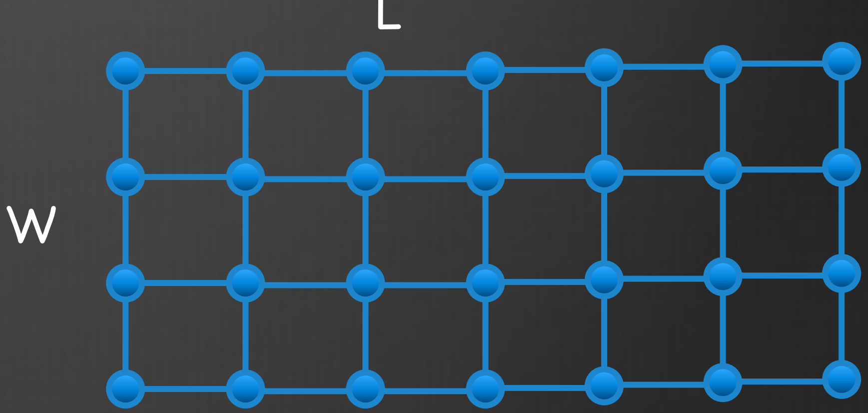


Photo credit:
Google Quantum AI Lab

How low can we make depth?

[Terhal-DiVincenzo '04] showed worst-case hardness of simulation as soon as T≥3. (T=2 is easy.)

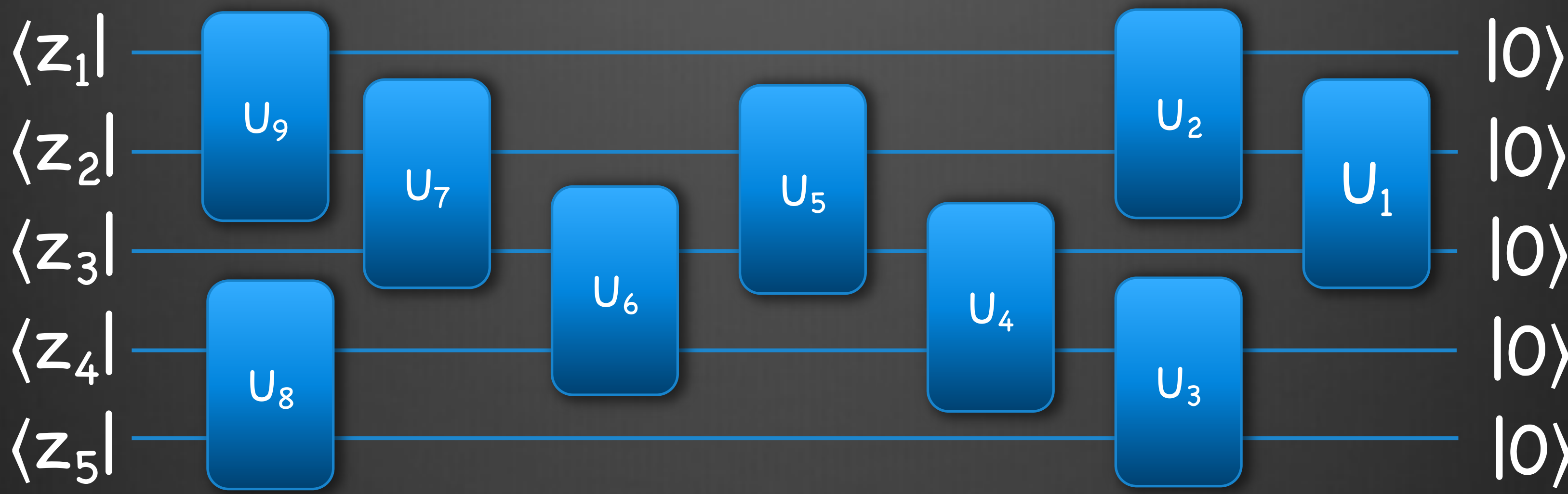
measurement-based quantum computing (MBQC)



- prepare L x W cluster state in O(1) depth
- single-qubit measurements simulate depth-W circuit on line of L qubits
- implies classical hardness is ≥ exp(min(L,W)).
- tensor contraction achieves this

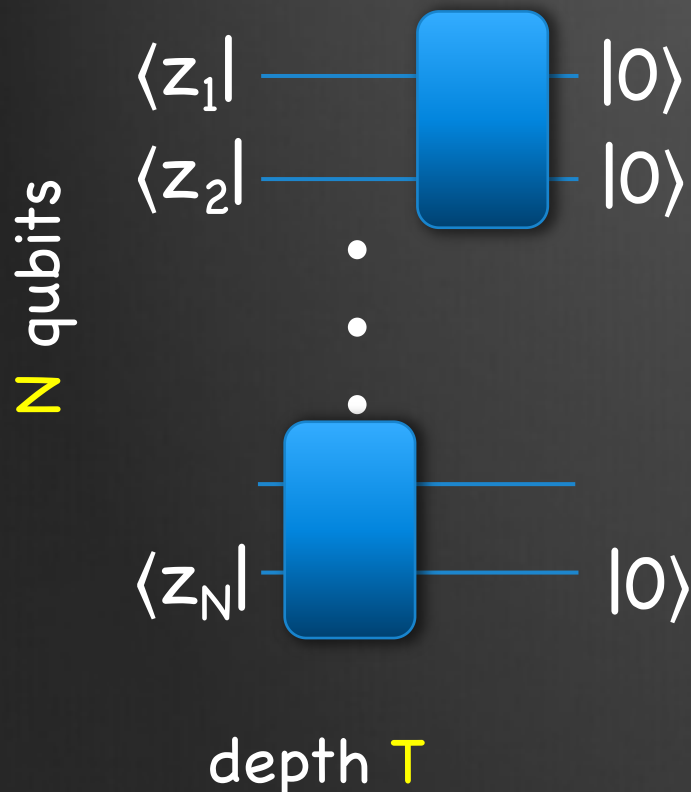
tensor contraction

$\langle z_1 z_2 z_3 z_4 z_5 | \, U_9 U_8 U_7 U_6 U_5 U_4 U_3 U_2 U_1 \, |00000\rangle =$

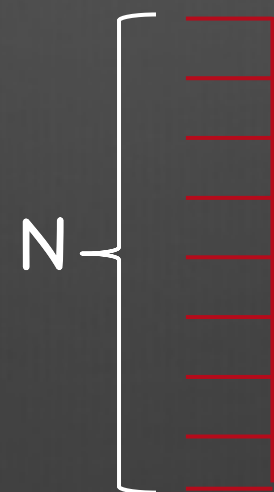


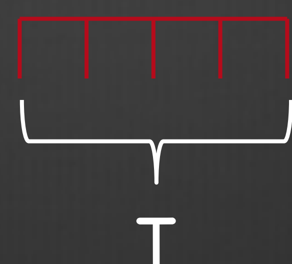
= tensor with 4 indices, each dim 2

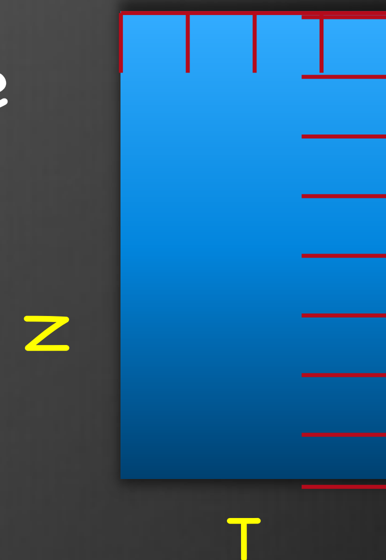
tensor contraction in 1-D



intermediate tensors can be



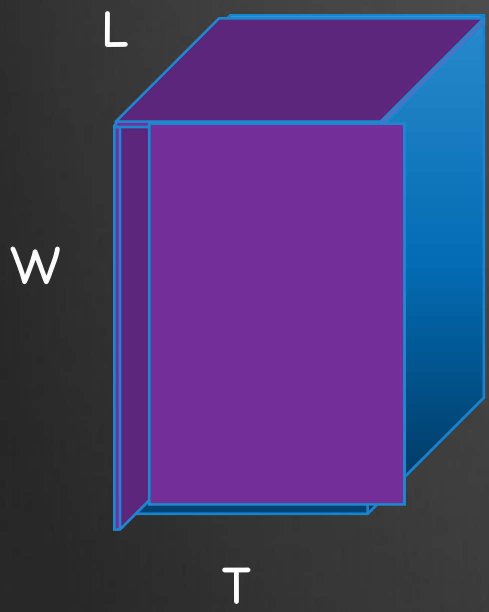




run time:
T exp(N) or
N exp(T)

[Napp, La Placa, Dalzell, Brandão, Harrow, in preparation]

simulating 2-D circuits



can be simulated in time 2^{LW} or 2^{LT} or 2^{WT}

Depth T=O(1) circuit on $\sqrt{N} \times \sqrt{N}$ grid

Naively takes time $2^{O(\sqrt{N})}$
≈ $\sqrt{N}$ qubits on line for time $\sqrt{N}$.

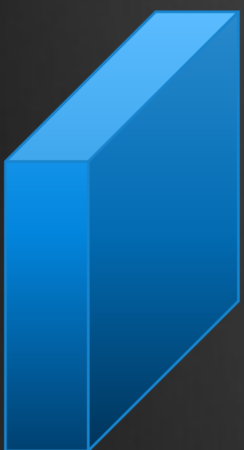
But 1-D effective evolution is not unitary.

Entanglement has phase transition from area law -> volume law.

T=3 in area law phase →
$N^{O(1)}$-time classical simulation for approximate sampling of random circuits.
Exact or worst-case is #P-hard.

cheaper tensor contraction

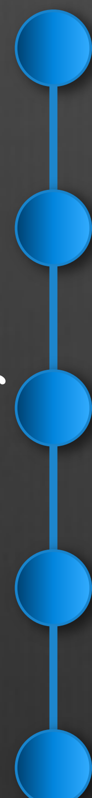
T=O(1)
$\sqrt{N} \times \sqrt{N}$ grid



effective
time
evolution

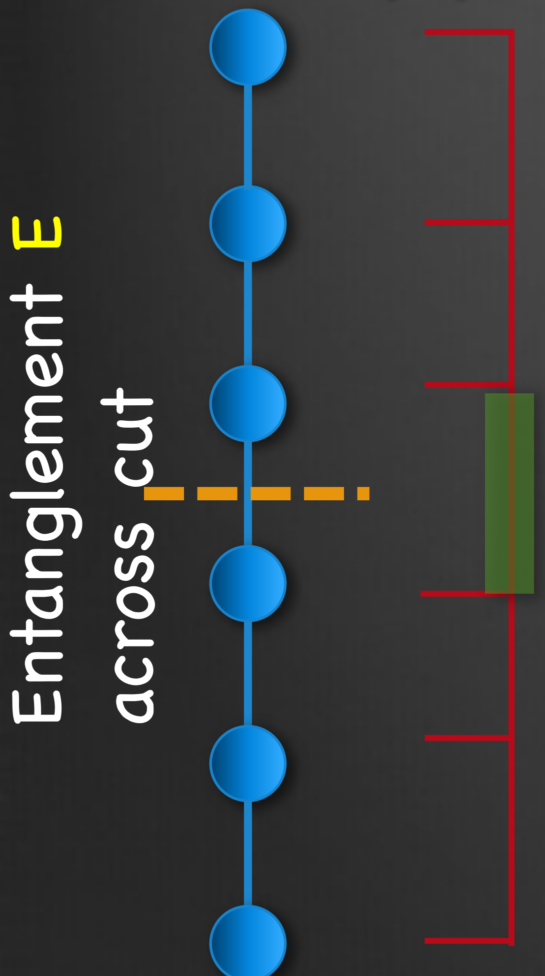
weak
measurement

$\sqrt{N}$ qubits
evolving for
$\sqrt{N}$ time



sideways gates
generically not
unitary

Approximate simulation



Matrix product state

bond dimension exp(E)

Example:

$$\sum_k |k\rangle|\alpha_k\rangle$$

$$\sum_k \lambda_k \langle k|\langle k|$$

$$\sum_k |k\rangle|\beta_k\rangle$$

$$|\psi\rangle = \sum_{i,j} c_{i,j}|i\rangle \otimes |j\rangle$$

$$= \sum_k \lambda_k |\alpha_k\rangle|\beta_k\rangle$$

Simulation algorithm:

- Do tensor contraction
- Truncate bonds to dim exp(O(E)).

Run-time is $N2^{O(E)}$.

Does the algorithm work?

"Beware of bugs in the above code; I have only proved it correct, not tried it."
--Donald Knuth

1. Yes.
   We tested it and simulated 400x400 grids on a laptop.

2. Probably.
   We proved a phase transition in something like the effective entanglement.

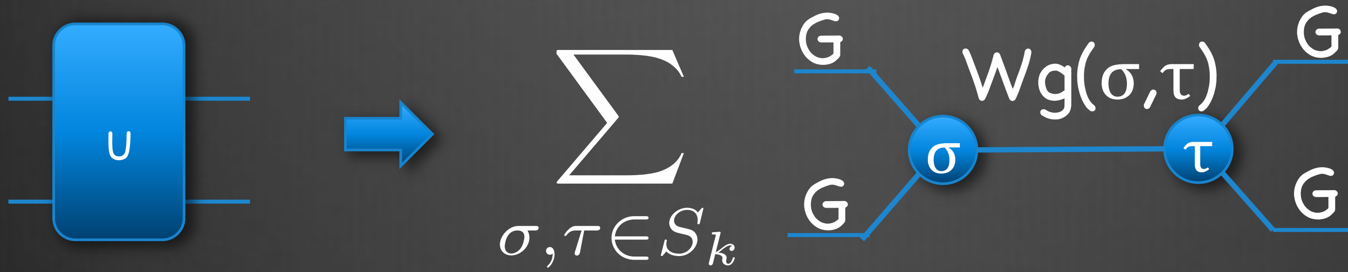
3. Sometimes.
   The extended brickwork architecture is #P-hard to simulate exactly but our algorithm is proven to work on it.

stat mech model

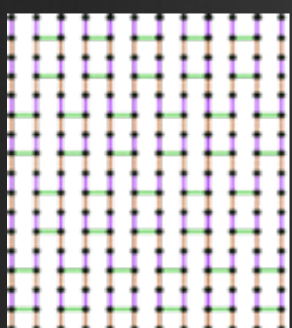
Qubits → dim-q particles.

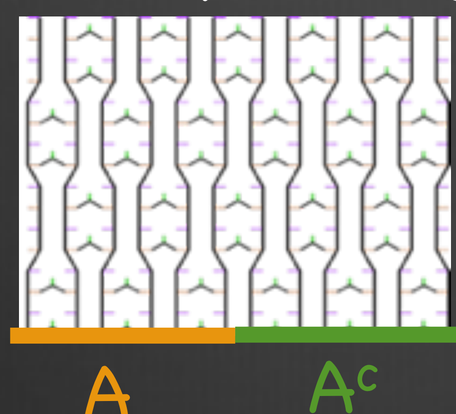
$E[\mathrm{tr}[\rho_A^k]]$ = partition function

$Wg^{-1}(\sigma,\tau)$
$= G(\sigma,\tau)$
$= q^{-\mathrm{dist}(\sigma,\tau)}$

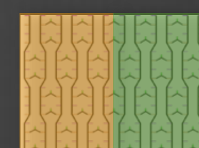


k=2 ⟷ anisotropic Ising

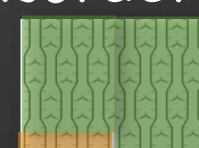




ordered



disordered



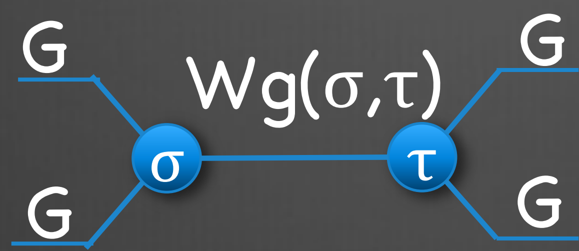
volume law
$E = O(\sqrt{N})$
≫1 ~ holography
q_c = 3.249...
1+ε ~ high-T Potts
area law
$E = O(1)$
q

conjecture: decimating yields nonnegative weights for all k, q

random tensor networks

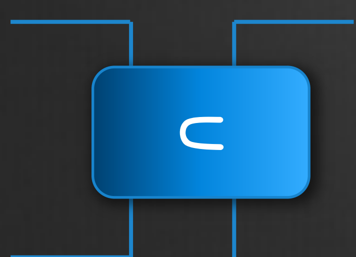
q = local dim, $E[\mathrm{tr}[\rho^k]]$

$$\sum_{\sigma,\tau \in S_k}$$



$Wg^{-1}(\sigma,\tau)$
$= G(\sigma,\tau)$
$= q^{-\mathrm{dist}(\sigma,\,\tau)}$

- $E_{U\sim \mathrm{Haar}}[U^{\otimes k} \otimes (U^*)^{\otimes k}] = \Sigma_{\sigma,\tau}\, Wg(\sigma,\tau)\, |\sigma\rangle\langle\tau|$
- $E_{G\sim \mathrm{GUE}}[G^{\otimes k} \otimes (G^*)^{\otimes k}] = \Sigma_{\sigma}\, |\sigma\rangle\langle\sigma|$
- $\langle\sigma|\tau\rangle = G(\sigma,\tau)$
- $\|\, G - I \,\|_{op}\, , \|\, Wg - I \,\|_{op} \leq k^2 / q$



$\approx$

$k^2 \ll q$

Open questions

- Rigorously prove the location of the phase transition and the correctness of the algorithm.
- Random tensor networks with low bond dimension
- Universality classes in random circuits?
- (time-independent) Hamiltonian versions?
- Where exactly is the boundary between easy and hard?

Thanks!

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Alex Dalzell

Rolando La Placa

John Napp

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