Quantum Speedup for Graph Sparsification, Cut Approximation and Laplacian Solving

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Simons Institute, April 2020

(arXiv:1911.07306)

Graphs

graphs are nice

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- all over computer science, discrete math, biology, . . .

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- describe relations, networks, groups, . . .

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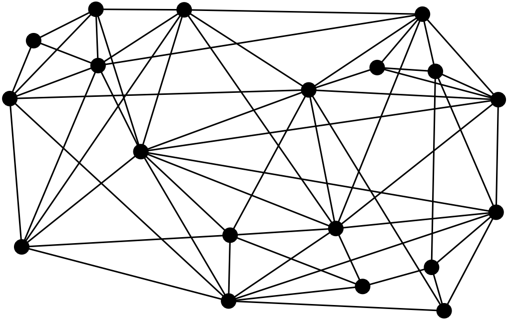
can we **compress** general graphs to sparse graphs ?

Graph Sparsification

undirected, weighted graph $G = (V, E, w)$

n nodes and m edges, $m \leq \binom{n}{2}$



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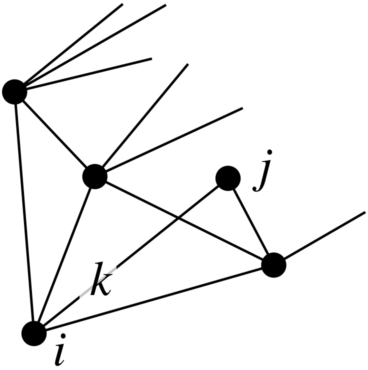


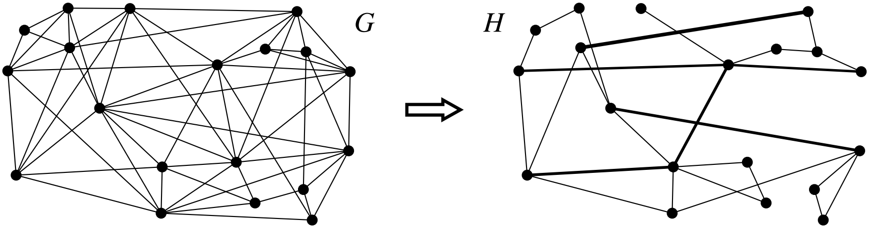
adjacency-list access

query (i, k) returns k-th neighbor j of node i

Graph Sparsification

"graph sparsification"

= reduce number of edges, while preserving interesting quantities



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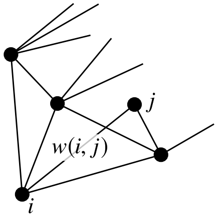

$$L_G = D - A$$

with

$$(D)_{ii} = \sum_j w(i,j) \quad \text{and} \quad (A)_{ij} = w(i,j)$$

Graph Laplacian

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$$L_G = \sum_{(i,j) \in E} w(i,j)\, L_{(i,j)}$$

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with

$$L_{(i,j)} = (e_i - e_j)\,(e_i - e_j)^T = \begin{bmatrix} 0 & & \dots & & 0 \\ \vdots & & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{(i,j)} & & \vdots \\ 0 & & \dots & & 0 \end{bmatrix}$$

Graph Laplacian

mainly interested in **quadratic forms in L_G**

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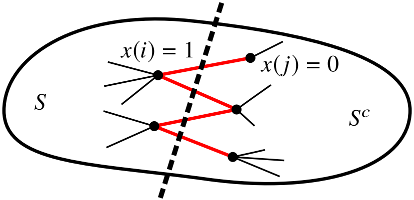

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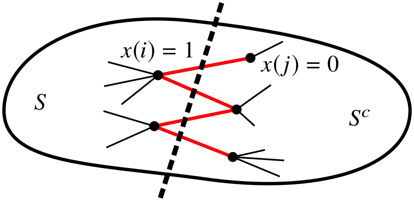

$x_S^T L_G x_S$

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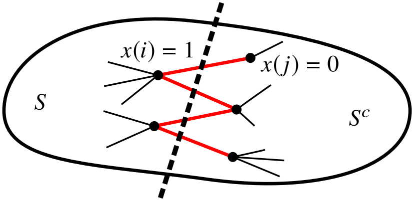

$$x_S^T L_G x_S = \sum_{(i,j)} w(i,j)(x_S(i) - x_S(j))^2$$

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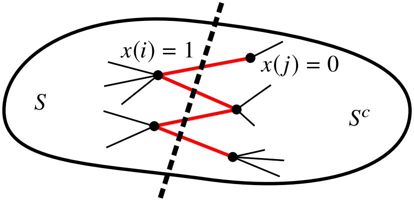

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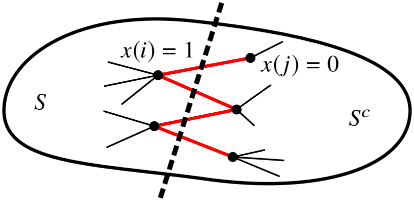

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as it turns out,

quadratic forms

$$x^T L_G x \quad \text{and} \quad x^T L_G^+ x \quad \text{for } x \in \mathbb{R}^n$$

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$\rightarrow$ interested in preserving quadratic forms!

Spectral Sparsification

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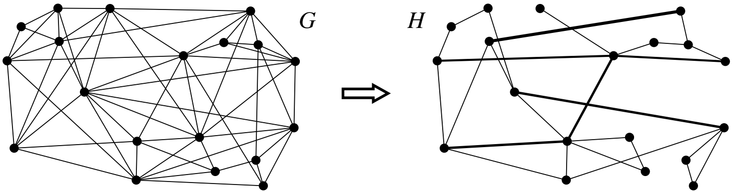
= approximately **preserve all quadratic forms**



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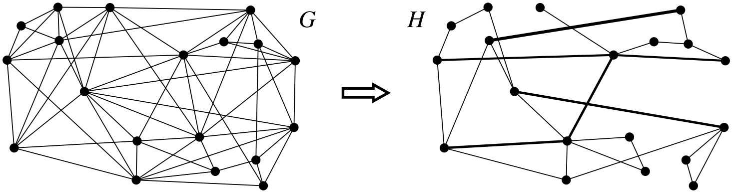

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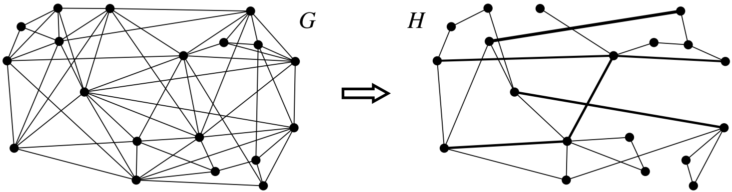

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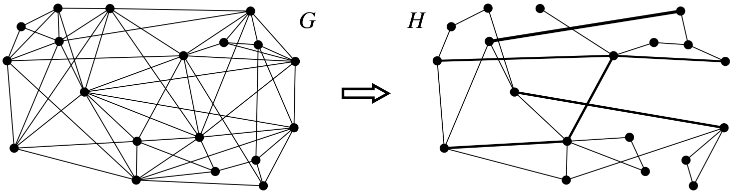

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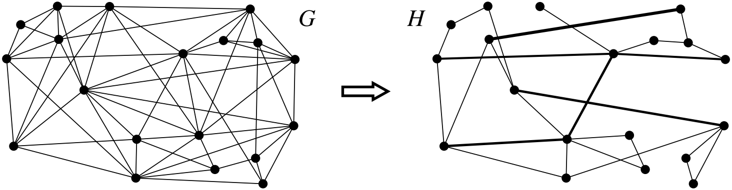

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$$(1-\epsilon)\, L_G \preceq L_H \preceq (1+\epsilon)\, L_G$$

Spectral Sparsification

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(only relevant when $\epsilon \leq \sqrt{n/m}$)

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- max cut (Arora-Kale '16)
- min cut (Karger '00)
- min st-cut (Peng '16)
- sparsest cut (Sherman '09)
- ...

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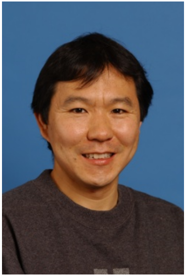
= Gödel prize 2015



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$\widetilde{O}(m)$ approximation algorithms for

- electrical flows and max flows
- spectral clustering
- random walk properties
- learning from data on graphs
- ...

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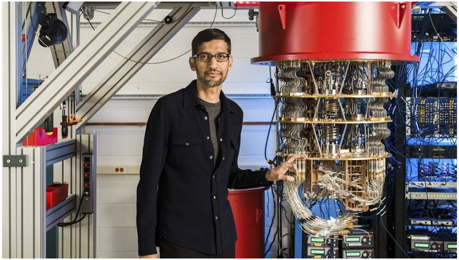

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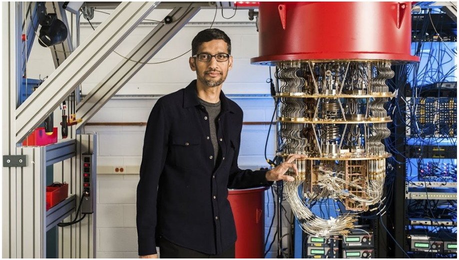

(disclaimer: not with this one we won't)

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Sparsification by edge sampling:

1. associate probabilities $\{p_e\}$ to every edge
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[Spielman-Srivastava '08]:
give high p_e to edges with high **effective resistance!**

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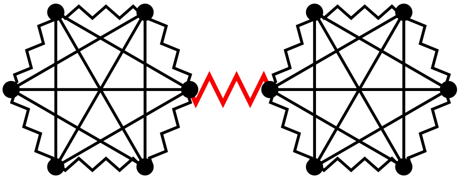

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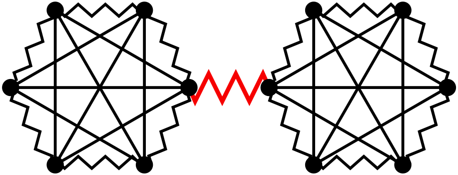

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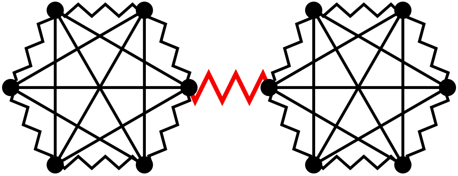

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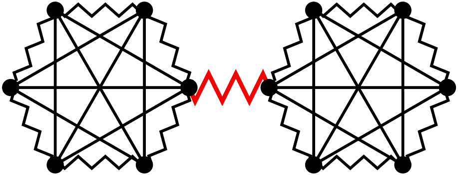

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$\rightarrow$ small if many short and parallel paths from i to j !

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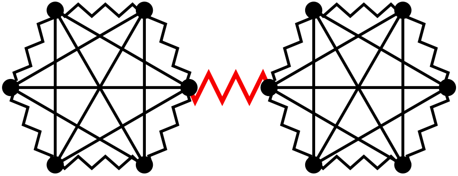

effective resistance $R_{(i,j)}$

red edge: $R_e = 1$

black edges: $R_e \in O(1/n)$

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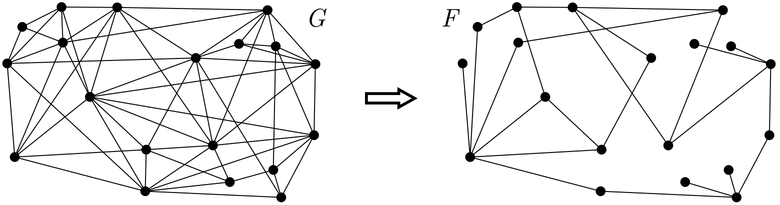

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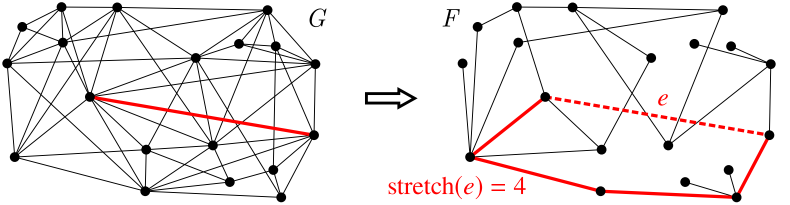

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proof idea for $R_e = 1$*:*

- if $R_e = 1$, there are no alternative paths between endpoints
- hence, e must be present in spanner

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Iterative sparsification:

1. construct $\widetilde{O}(1/\epsilon^2)$ spanners and keep these edges
2. keep any remaining edge with probability $1/2$, and double its weight

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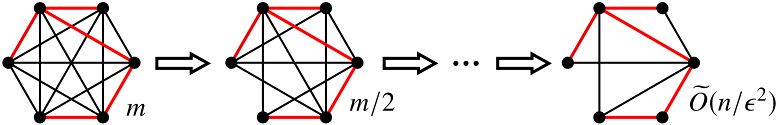
$\rightarrow$ *repeat $O(\log n)$ times:* ϵ-spectral sparsifier with $\widetilde{O}(n/\epsilon^2)$ edges



Quantum Sparsification Algorithm

Quantum Sparsification Algorithm

= quantum spanner algorithm

+ k-independent oracle

+ a magic trick

Quantum Spanner Algorithm

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Theorem ("easy")

There is a quantum spanner algorithm with **query complexity**

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$\rightarrow$ can prove: $\widetilde{O}(n)$ edges are found using $\widetilde{O}(\sqrt{mn})$ queries

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[Dürr-Heiligman-Høyer-Mhalla '04]

quantum speedup for constructing SPTs

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1. use quantum algorithm to construct $\widetilde{O}(1/\epsilon^2)$ spanners, keep these edges
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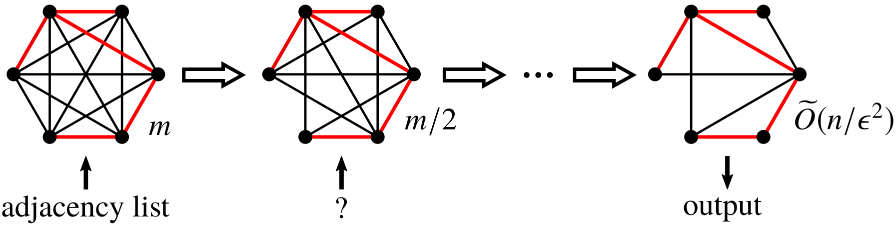

Query Access to Random String

maintain (offline) random string $x \in \{0,1\}^{\binom{n}{2}}$

1	0	0	1	1	0	1	1	1	0	1	0	0

edge (i,j) discarded   edge (i',j') kept

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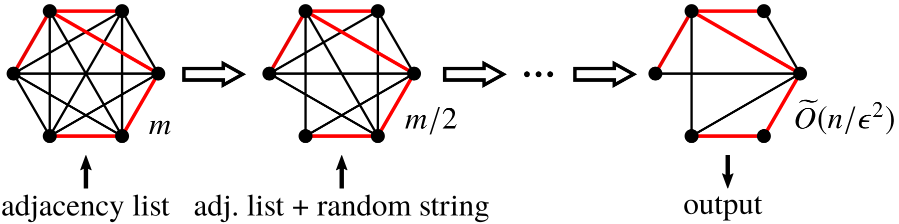

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- quantum this is not possible: can address all bits in superposition

Rid of Random String

luckily, we can outsmart this quantum demon:

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Fact

*$k/2$-query quantum algorithm cannot distinguish uniformly random string from k-wise independent string **

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* k-wise independent string $x \in \{0,1\}^{\binom{n}{2}}$
behaves uniformly random on every subset of k bits

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aim for quantum algorithm making $\sim \sqrt{mn}$ queries,
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Theorem (Christiani-Pagh-Thorup ’15)

Can construct in preprocessing time $\widetilde{O}(k)$ a k-independent oracle that simulates queries to k-wise independent string in time $\widetilde{O}(1)$ per query.

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Corollary

Any k-query quantum algorithm that queries a uniformly random string can be simulated in time $\widetilde{O}(k)$ without random string.

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Theorem

There is a quantum algorithm that constructs an ϵ-spectral sparsifier with $\widetilde{O}(n/\epsilon^2)$ edges in time

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A Magic Trick

Münchhausen O. Herrfurth pinx

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* assuming $\epsilon \geq \sqrt{n/m}$, it holds that $\widetilde{O}(\sqrt{mn}/\epsilon) \in \widetilde{O}(m)$

this work:

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2. matching $\widetilde{\Omega}(\sqrt{mn}/\epsilon)$ **lower bound**

3. applications: quantum speedup for
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“hence”

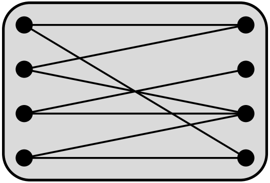
finding $\widetilde{O}(n/\epsilon^2)$ edges of sparsifier among m edges takes time

$\widetilde{\Omega}(\sqrt{mn}/\epsilon)$

Unsparsifiable Graph

Unsparsifiable Graph

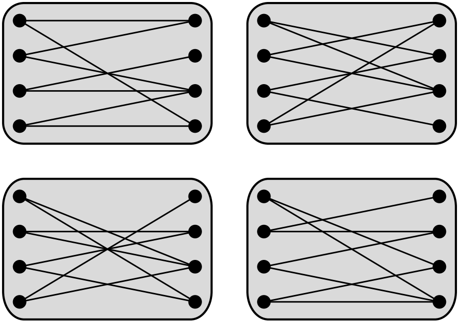
random bipartite graph on $1/\epsilon^2$ nodes



Unsparsifiable Graph

$\epsilon^2 n$ copies

= random graph $H(n, \epsilon)$ with n nodes and $O(n/\epsilon^2)$ edges



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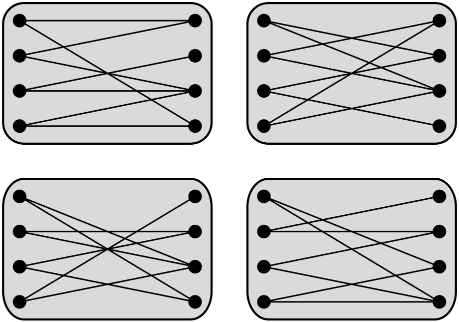

Theorem (Andoni-Chen-Krauthgamer-Qin-Woodruff-Zhang '16)

Any ϵ-spectral sparsifier of $H(n, \epsilon)$ must contain a constant fraction of its edges.

Hiding a Sparsifier

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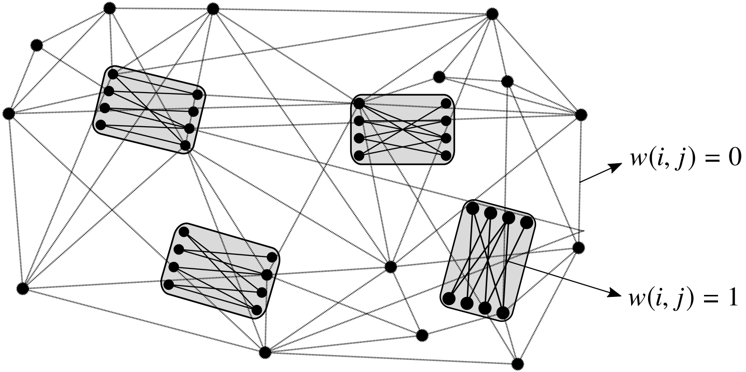
given n, m, ϵ:

we "hide" $H(n, \epsilon)$ in larger $G(n, m, \epsilon)$ with n nodes and m edges



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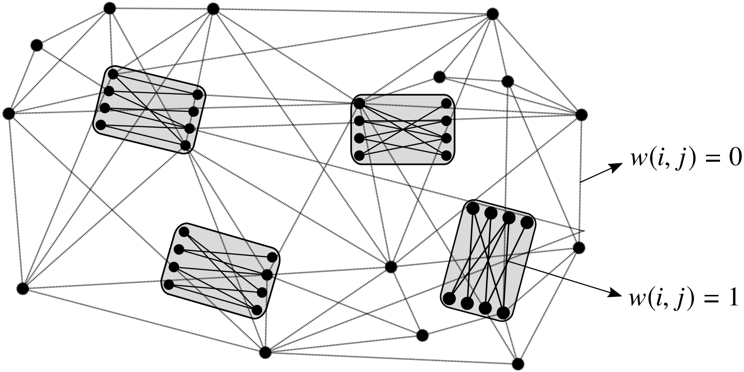

$\rightarrow$ ϵ-spectral sparsifier of $G(n, m, \epsilon)$ must find constant fraction of $H(n, \epsilon)$

Proving a Lower Bound

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"hidden" copy of random graph:
every edge of sparsifier is **hidden** among $N = m/(n\epsilon^2)$ entries

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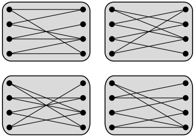

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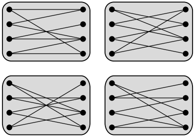

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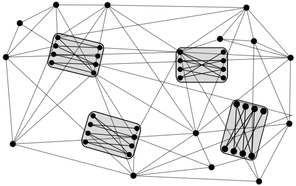

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= relational problem composed with OR_N

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Theorem (proof by A. Belov and T. Lee, to be published)

The quantum query complexity of an efficiently verifiable relational problem, with lower bound L, composed with the OR_N-function, is

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for $L = \widetilde{\Omega}(n)$ and $N = m/(n\epsilon^2)$:

Corollary

The quantum query complexity of explicity outputting an ϵ-spectral sparsifier of a graph with n nodes and m edges is

$$\widetilde{\Omega}(\sqrt{mn}/\epsilon).$$

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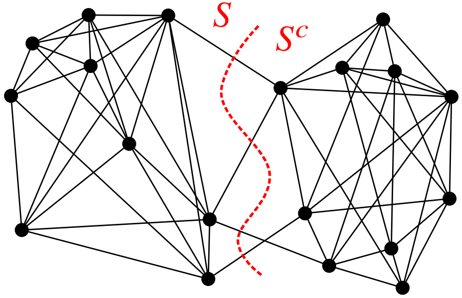
=

approximate $\widetilde{O}(\sqrt{mn}/\epsilon)$ **quantum algorithm** for P

Cut Approximation

MIN CUT:

find cut (S, S^c) that minimizes cut value $\mathrm{cut}_G(S)$



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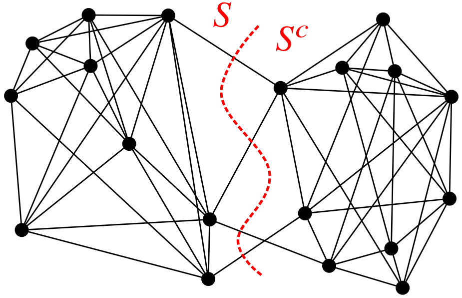
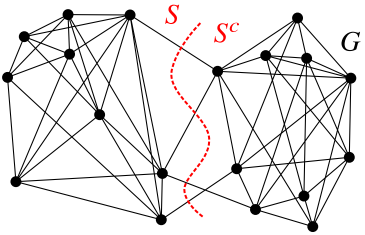


classically: can find MIN CUT in time $\widetilde{O}(m)$ (Karger '00)

Cut Approximation

MIN CUT of ϵ-spectral sparsifier H
gives ϵ-approximation of MIN CUT of G



Cut Approximation

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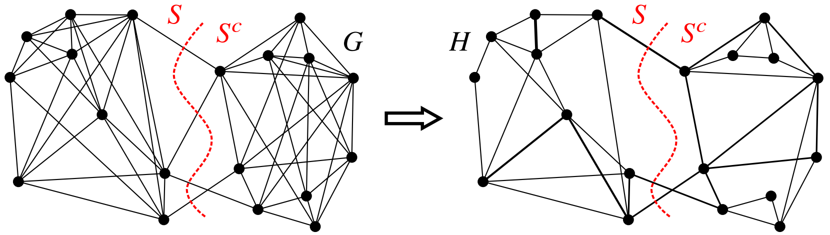

quantum sparsify G to H in $\widetilde{O}(\sqrt{mn}/\epsilon)$
+ **classical** MIN CUT on H in $\widetilde{O}(n/\epsilon^2)$ (Karger '00)

= $\widetilde{O}(\sqrt{mn}/\epsilon)$ **quantum** algorithm for ϵ-MIN CUT

Cut Approximation

	Classical	Quantum (this work)
ϵ-MIN CUT	$\widetilde{O}(m)$ (Karger'00)	$\widetilde{O}(\sqrt{mn}/\epsilon)$
ϵ-MIN st-CUT	$\widetilde{O}(m + n/\epsilon^5)$ (Peng'16)	$\widetilde{O}(\sqrt{mn}/\epsilon + n/\epsilon^5)$
$\sqrt{\log n}$-SPARSEST CUT/ -BAL. SEPARATOR	$\widetilde{O}(m + n^{1+\delta})$ (Sherman'09)	$\widetilde{O}(\sqrt{mn} + n^{1+\delta})$
.878-MAX CUT	$\widetilde{O}(m)$ (Arora-Kale'07)	$\widetilde{O}(\sqrt{mn})$

Laplacian Solving

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general linear system $Ax = b$

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given A and b, with $nnz(A) = m$,

complexity of approximating x is $\widetilde{O}(\min\{mn, n^{\omega}\})$ ($\omega < 2.373$)

Laplacian Solving

Laplacian system $Lx = b$

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given L and b, with $nnz(L) = m$,

complexity of approximating x is $\widetilde{O}(m)$ [Spielman-Teng '04]

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if H sparsifier of G then $L_H^+ b \approx L_G^+ b$

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quantum algorithm to sparsify G to H in $\widetilde{O}(\sqrt{mn}/\epsilon)$

+ solve $L_H x = b$ **classically** in $\widetilde{O}(n/\epsilon^2)$

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quantum algorithm for Laplacian solving in $\widetilde{O}(\sqrt{mn}/\epsilon)$

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quantum algorithm for Laplacian solving in $\widetilde{O}(\sqrt{mn}/\epsilon)$

(+ quantum reduction for symmetric, diagonally dominant systems)

Laplacian Solving and Friends

	Classical	Quantum (this work)
ϵ-SDD Solving	$\widetilde{O}(m)$ (ST'04)	$\widetilde{O}(\sqrt{mn}/\epsilon)$
ϵ-Effective Resistance (single)	$\widetilde{O}(m)$	$\widetilde{O}(\sqrt{mn}/\epsilon)$ prior: $\widetilde{O}(\sqrt{mn}/\epsilon^2)$
ϵ-Effective Resistance (all)	$\widetilde{O}(m+n/\epsilon^4)$ (Spielman-Srivastava'08)	$\widetilde{O}(\sqrt{mn}/\epsilon+n/\epsilon^4)$
$O(1)$-Cover Time	$\widetilde{O}(m)$ (Ding-Lee-Peres'10)	$\widetilde{O}(\sqrt{mn})$
k bottom eigenvalues	$\widetilde{O}(m+kn/\epsilon^2)$	$\widetilde{O}(\sqrt{mn}/\epsilon+kn/\epsilon^2)$ prior, $k=1$: $\widetilde{O}(n^2/\epsilon)$
Spectral k-means clustering	$\widetilde{O}(m+n\,\mathrm{poly}(k))$	$\widetilde{O}(\sqrt{mn}+n\,\mathrm{poly}(k))$

summary:

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- matching **lower bounds for applications?**
  e.g., $\Omega(\sqrt{mn}/\epsilon)$ for approximate min cut or Laplacian solving?

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  can we do **better for unweighted graphs?**

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thank you! stay safe!