Quantum Coupon Collector

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Collecting coupons

- Every year Albert Heijn hands out cards of soccer players ("voetbalplaatjes"). You get a random one for each 2.50€ you spend on groceries

- There are 18 teams, 11 players per team: 198 different cards
- Your nephew really wants to have a complete set.
  How much money do you need to spend to get each card?
- Obviously, at least $198 \times 2.5 = 495$€
- But it's worse: if you already have a copy of most of the cards, then your next 2.5€ spending will likely give you a card that you already have

Analysis of coupon collector

- Suppose you already have i out of k coupons, and you get another, uniformly random coupon.

$$\Pr[\text{see new coupon}] = \frac{k-i}{k}$$

- Total expected number of samples:

$$\sum_{i=0}^{k-1} \frac{k}{k-i} = k \sum_{j=1}^{k} \frac{1}{j} = k \ln(k) + \Theta(k)$$

- With $k = 198$, that's 1162 samples. Need to spend 2905€!

- You're unlikely to finish much earlier. Variance in number of samples is $\Theta(k^2)$, so typically you need $k \ln(k) \pm O(k)$ samples

Can quantum help somehow?

- Suppose get "quantum samples" instead of random samples. You want to learn unknown set $S \subseteq [n]$ of size k from states

$$|S\rangle = \frac{1}{\sqrt{k}} \sum_{i \in S} |i\rangle$$

- If you measure, you get a uniformly random sample from S, and we know $k \ln(k)$ random samples needed to learn S
- Maybe there's something smarter, using fewer copies of $|S\rangle$?
  - **yes** if the number of "missing items" $(m = n - k)$ is small
  - **no** otherwise

Learning S by sampling the missing elements

- $|S\rangle = \sqrt{\frac{k}{n}}|[n]\rangle + \sqrt{\frac{m}{n}}|\psi\rangle$

  where $|\psi\rangle = \sqrt{\frac{m}{n}}|S\rangle - \sqrt{\frac{k}{n}}|\overline{S}\rangle \approx -|\overline{S}\rangle$ if $m \ll n$
- If we measure one copy of $|S\rangle$ with 2-outcome measurement $|[n]\rangle\langle[n]|$ vs $I - |[n]\rangle\langle[n]|$, then with prob $\frac{m}{n}$ we obtain $|\psi\rangle$
- Thus we can convert an expected number of $\frac{n}{m}$ copies of $|S\rangle$ into one copy of $|\overline{S}\rangle$ (up to small errror)
- Now we can sample uniformly from the *complement* of S!

  $\overline{S}$ has m elements, so $O(m\log(m+1))$ copies of $|\overline{S}\rangle$ suffice.

  Hence $O(n\log(m+1))$ copies of $|S\rangle$ suffice for a "quantum coupon collector". For $m = O(1)$ and $k = n - O(1)$, this beats classical coupon collector by a log-factor

Matching lower bound on number of copies of $|S\rangle$

- Claim: you need $T = \Omega(k \log(m+1))$ copies of $|S\rangle$ to learn the k-set $S \subseteq [n]$ (assume $m \leq n/2$)

- Approach: Use the adversary lower bound (without queries!)

  A learner should do state transformation: $|S\rangle^{\otimes T} \mapsto S$

  Consider Gram matrix $M_{SS'} = (\langle S|S'\rangle)^T$; and $F_{SS'} = 1 - \delta_{SS'}$

  State transformation problem can be solved iff $\gamma_2(M \circ F)$ is small

- Can witness $\gamma_2(M \circ F) \geq 1/2$ via an adversary matrix:

$$\gamma_2(M \circ F) = \max_{\|\Gamma\| \leq 1} \| \Gamma \circ M \circ F \|$$

  How to construct such Γ? Note that M, F-entries only depend on $|S \cap S'|$, so we need math that respects this symmetry.

Using the Johnson association scheme

- Define Boolean matrices $A_0, \ldots, A_m$ of dimension $N = \binom{n}{k}$, with $(A_j)_{SS'} = 1$ iff $|S \cap S'| = k - j$
- $\exists$ pairwise-orthogonal projectors $E_0, \ldots, E_m$ spanning the same space: $A_i = \sum_{j=0}^{m} \underbrace{p_i(j)}_{\text{eigenvalues}} E_j, \quad E_j = \frac{1}{N} \sum_{i=0}^{m} \underbrace{q_j(i)}_{\text{dual eigenvalues}} A_i,$
  $$E_i \circ E_j = \frac{1}{N} \sum_{\ell=0}^{m} \underbrace{q_{i,j}(\ell)}_{\text{Krein parameters}} E_\ell.$$ These parameters are known.
- $M_{SS'}$ entries only depend on $|S \cap S'|$, so we can write M as linear combination of A_is and hence of E_js
- Adversary matrix $\Gamma = \sum_{j=0}^{m} \gamma_j E_j$, with $\gamma_0 = \cdots = \gamma_{m-1} = 1$, $\gamma_m \in [-1, 0]$. Ensures $\| \Gamma \| \leq 1$, and diag(Γ)=0 (so $\Gamma \circ F = \Gamma$)
- Complicated calculation involving Krein parameters (similar to classical coupon!): if $T \ll k \log(m+1)$ then $\| \Gamma \circ M \| \geq 1/2$

Relevance for proper vs improper PAC learning

- PAC learner $\mathcal{A}$ for a concept class $\mathcal{C} = \{f : [n] \to \{0,1\}\}$: given samples $(x, f(x))$, $x \sim D$, for unknown target concept $f \in \mathcal{C}$, find hypothesis $h : [n] \to \{0,1\}$ that is close to f:

$$\forall f \in \mathcal{C} \;\; \forall D : \Pr_{x \sim D}[f(x) \neq h(x)] \leq \varepsilon \text{ w.h.p.}$$

- Fundamental Thm: Required # of samples is $\Theta(\text{VCdim}(\mathcal{C})/\varepsilon)$
- $\mathcal{A}$ is called a proper learner if $h \in \mathcal{C}$
- Requiring $\mathcal{A}$ to be proper can increase sample complexity: $\exists \mathcal{C}$ where proper learner needs $\Theta(\text{VCdim}(\mathcal{C}) \log(1/\varepsilon)/\varepsilon)$ examples.

  Related to coupon collector with $m = 1$, $\varepsilon = 1/n$: $\mathcal{C} = \{f_S : [n] \to \{0,1\}$ is indicator of $S\}$
- $O(\text{VCdim}(\mathcal{C})/\varepsilon)$ quantum examples suffice for proper learner

What if you can also **reflect** through $|S\rangle$?

- If you can get copies of $|S\rangle$, then maybe you actually have a quantum machine to produce such copies? $U : |0\rangle \mapsto |S\rangle$
- Doing U and U^{-1} would allow you to reflect through $|S\rangle$! $R_S : |S\rangle \mapsto |S\rangle$, $R_S : |\psi\rangle \mapsto -|\psi\rangle$ whenever $\langle\psi|S\rangle = 0$
- Finding S more quickly, using amplitude amplification:
  1. Start from $|[n]\rangle$, rotate to $|\overline{S}\rangle$. Measure, get $i_1 \in \overline{S}$
  2. Start from $|[n]\rangle$, rotate to $|\overline{S}\backslash\{i_1\}\rangle$. Measure, get $i_2 \in \overline{S}$

     $\vdots$

  Cost of finding all elements of $\overline{S}$: $\displaystyle\sum_{j=0}^{m-1} \sqrt{\frac{n-j}{m-j}} = O(\sqrt{km})$
- We show that this is tight for $m \leq n/2$, and also get tight bound $\Theta(k)$ for case $m \geq n/2$

Summary

- Classical coupon collector:
  learn k-set $S \subseteq [n]$ from $\Theta(k \log k)$ uniform samples

- Quantum coupon collector:
  learn k-set $S \subseteq [n]$ from $\Theta(k \log(m+1))$ uniform superpositions ($m = n - k$ is number of missing items)

- We also gave tight bounds for learning S from copies of $|S\rangle$ and reflections through $|S\rangle$

- Open problem: are the quantum sample complexities of proper and improper learning the same for all $\mathcal{C}$?