o=true; for(int z=0;o;z++) o=false; for(int j=0;j<=

Algebraic techniques for Algebraic lattices

Thomas Espitau

March 2, 2020, Simons Institute

NTT

Lattices and LLL

What is this all about?

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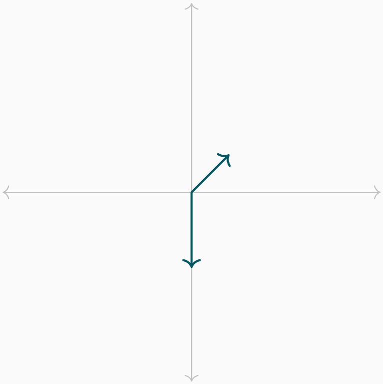

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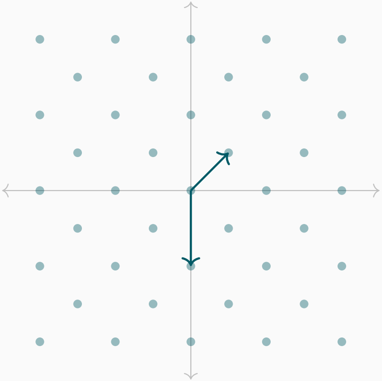

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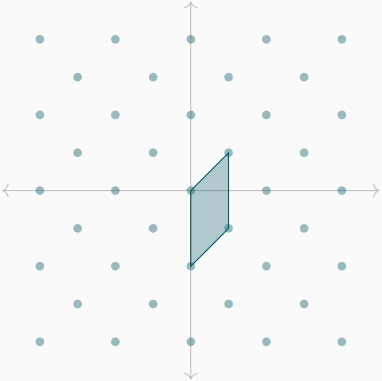

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Corresponds to the volume of the fundamental domain $\{\sum x_i v_i | 0 \leqslant x_i < 1\}$.

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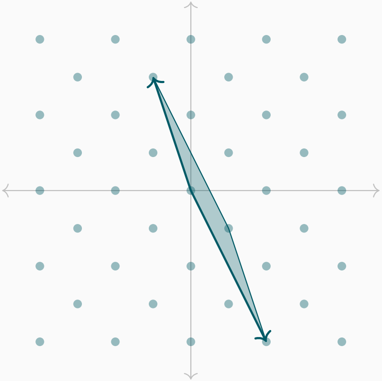

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Independent of the basis

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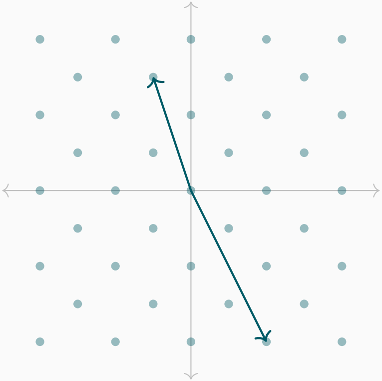

How to get a shorter basis?

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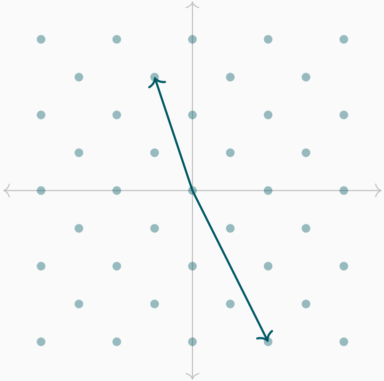

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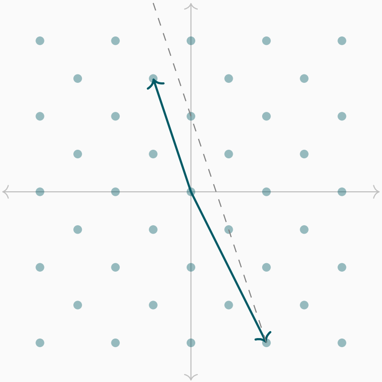

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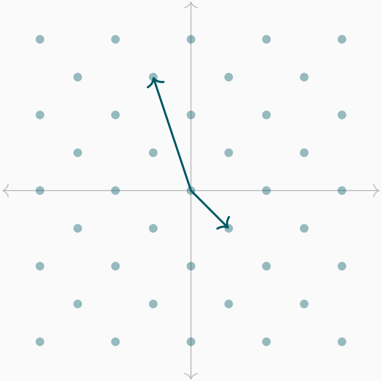

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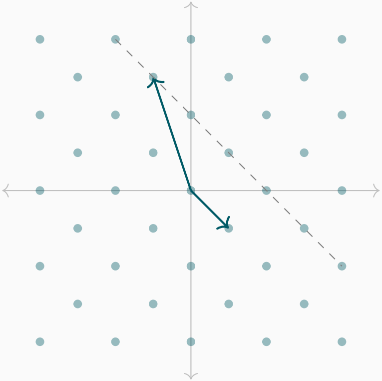

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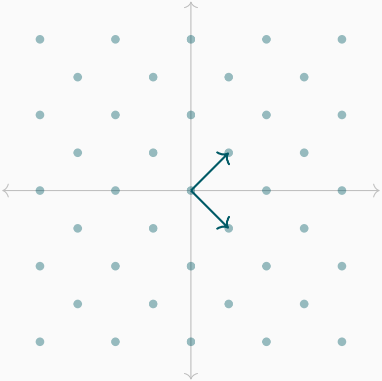

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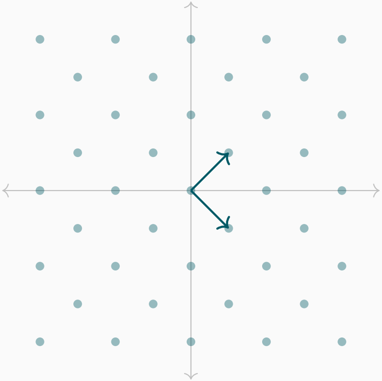

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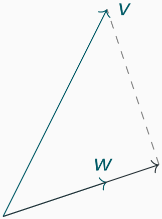

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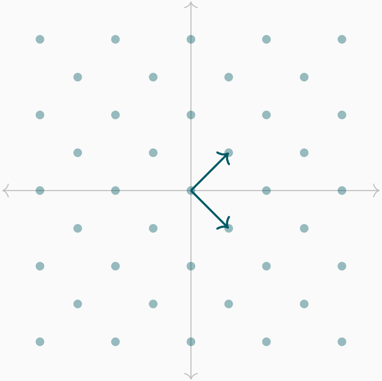

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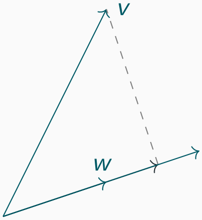

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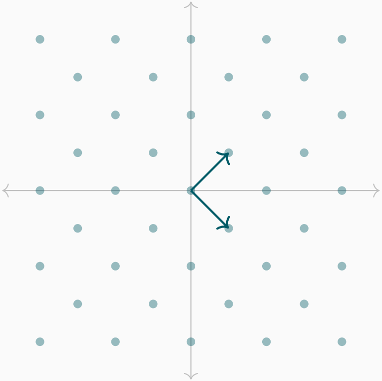

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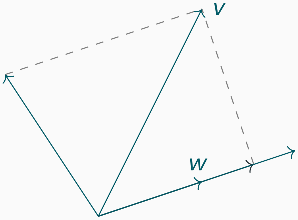

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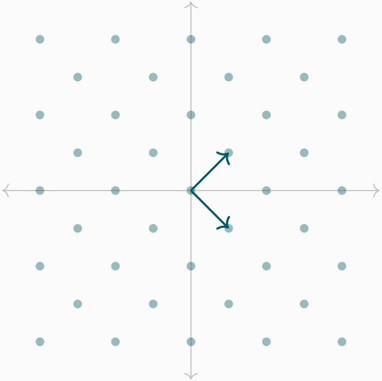

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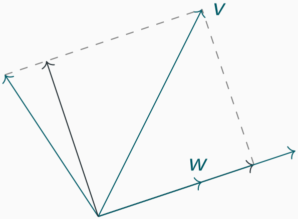

In dim 2... The Lagrange-Gauss reduction algorithm

Gauss-Lagrange reduction

if $\|v\| < \|u\|$ **then return Gauss**(v, u);

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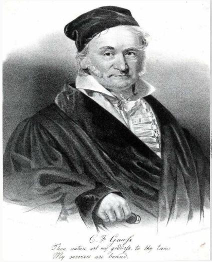

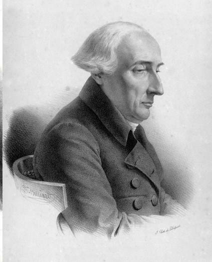

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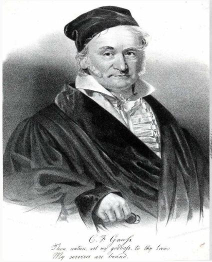

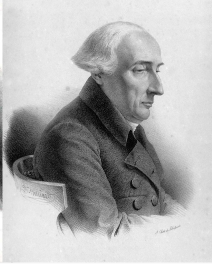

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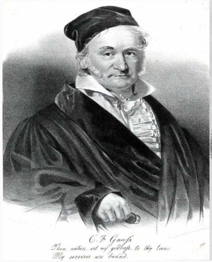

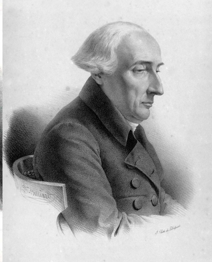

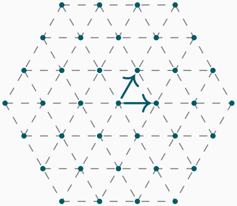

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- $\|u\|^2 \leqslant (4/3)\,\mathrm{covol}(\Lambda)$

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Finding a shortest/closest vector in a lattice is **hard**

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[LLL82] *There exists a* ***polynomial-time algorithm****, which given any lattice Λ, produces a vector in Λ of Euclidean length* ***at most*** *a factor of 2^d longer than a shortest vector.*

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- Computations in algebraic number theory (ideal computations, HNF, control of size of elements...)

Towards a polynomial-time reduction algorithm

Any basis $(v_1, \ldots, v_d)$ of a lattice Λ yields a filtration:

$$\{0\} = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{i-1} \subset \Lambda_i \subset \Lambda_{i+1} \subset \cdots \subset \Lambda_d = \Lambda$$

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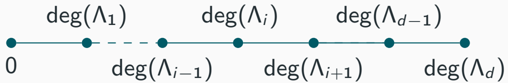

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Reduce the projected lattice with **Gauss algorithm**, **lift** and **replace**.

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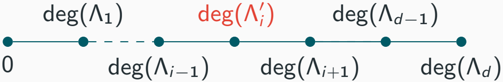

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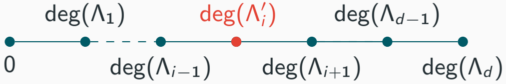

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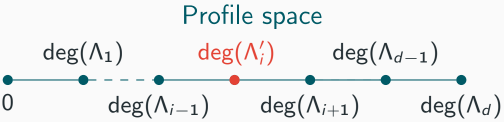

$$\deg(\Lambda'_i) \leqslant \deg(\Lambda_i)$$

Gauss's reduction is a *local* tool for densifying the filtration

From local to global: an iterative strategy

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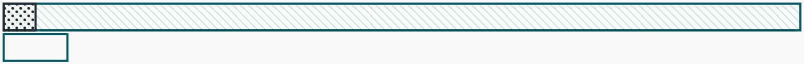

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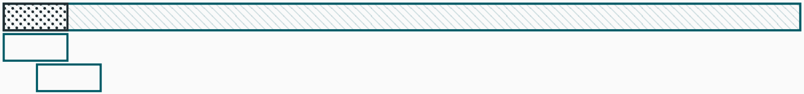

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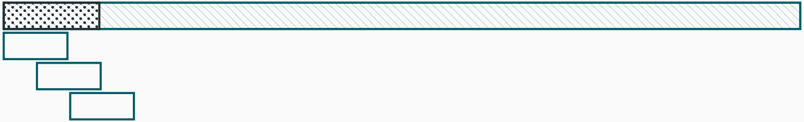

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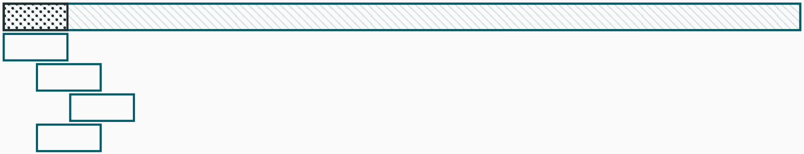

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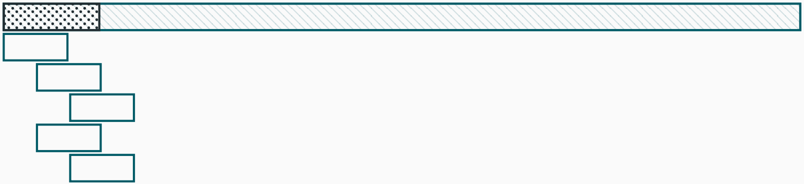

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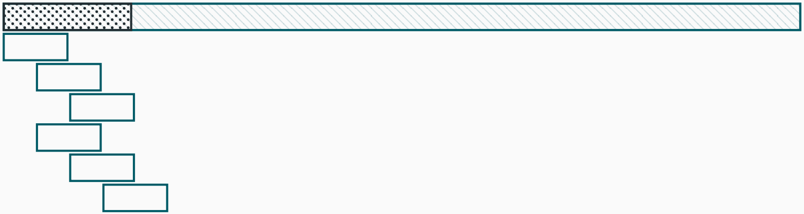

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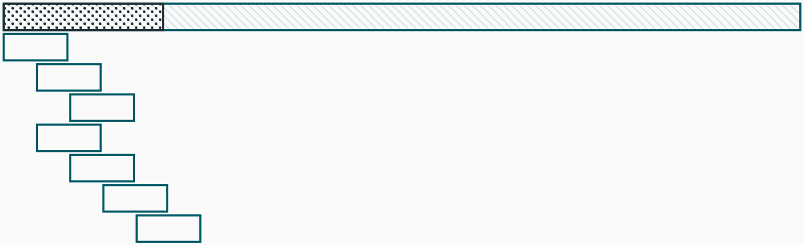

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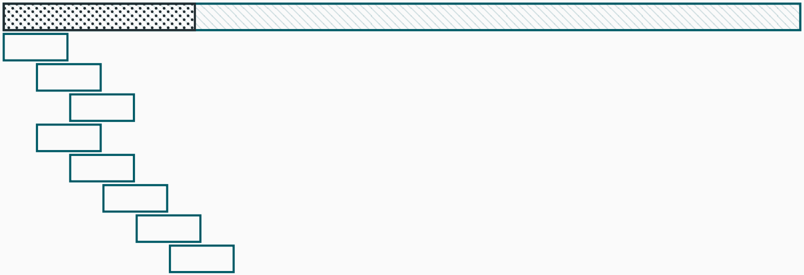

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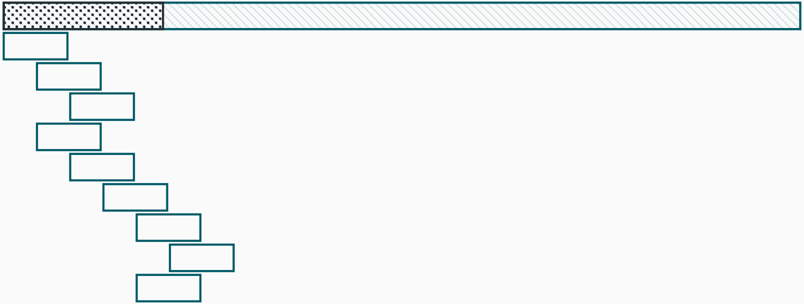

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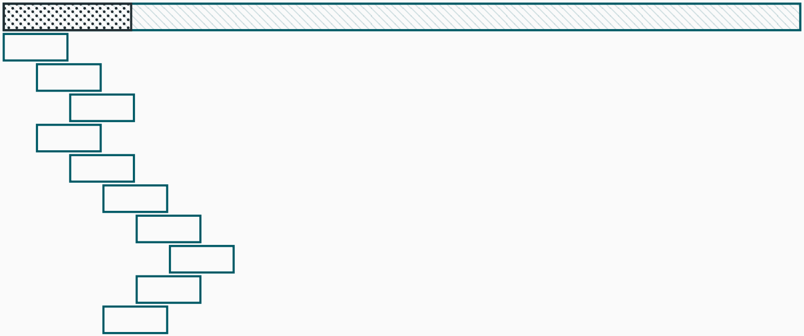

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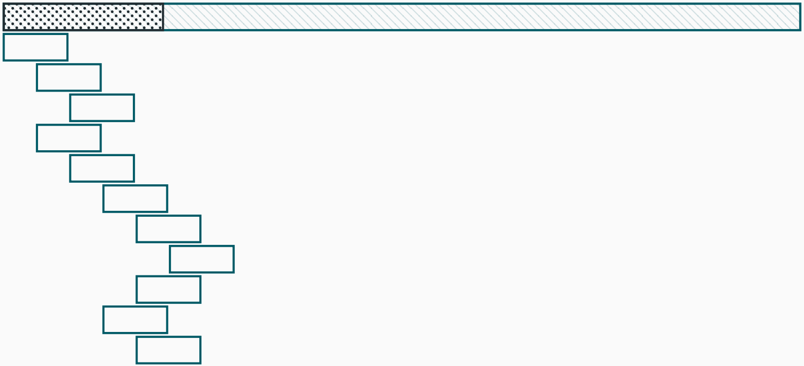

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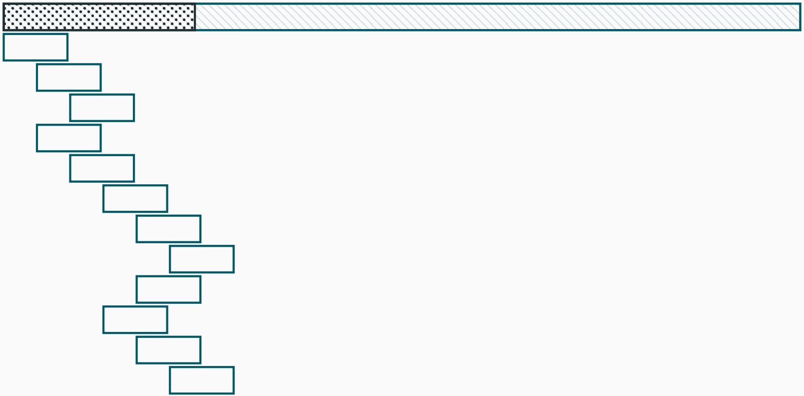

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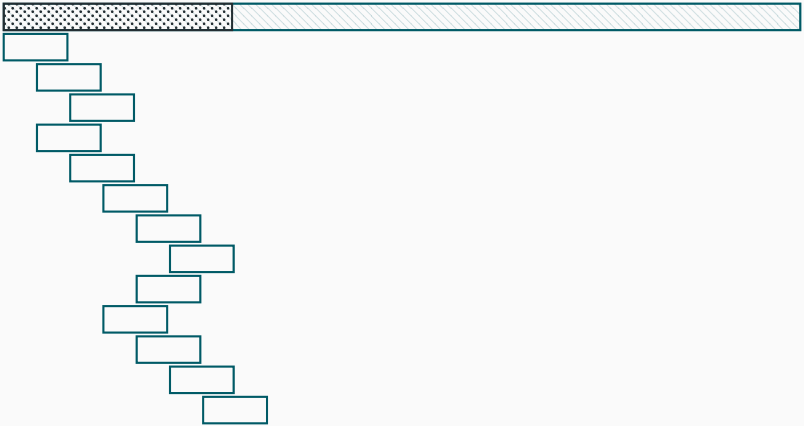

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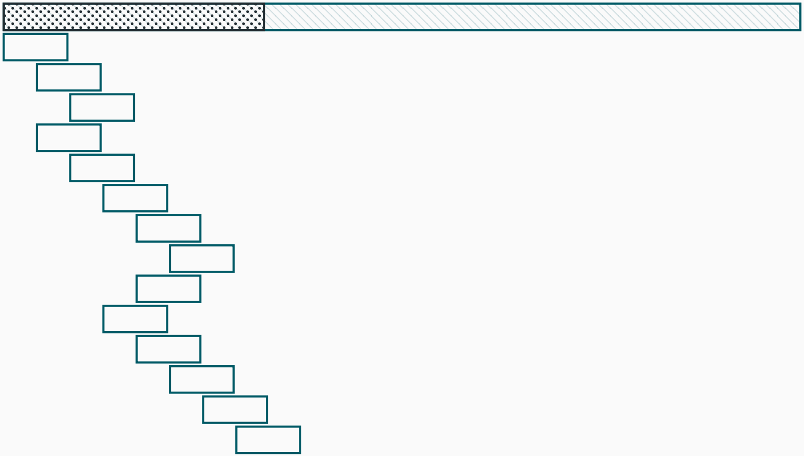

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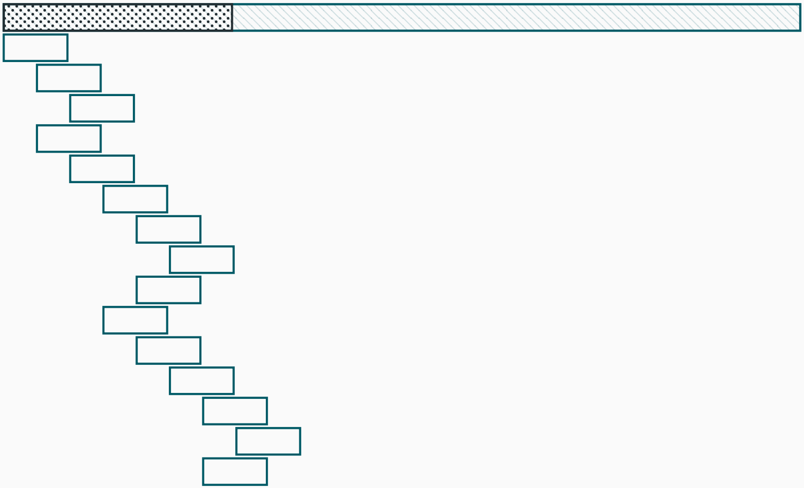

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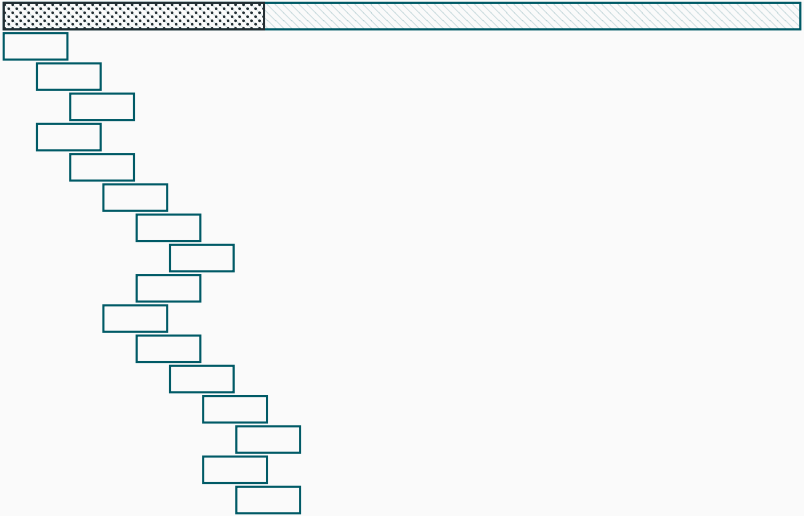

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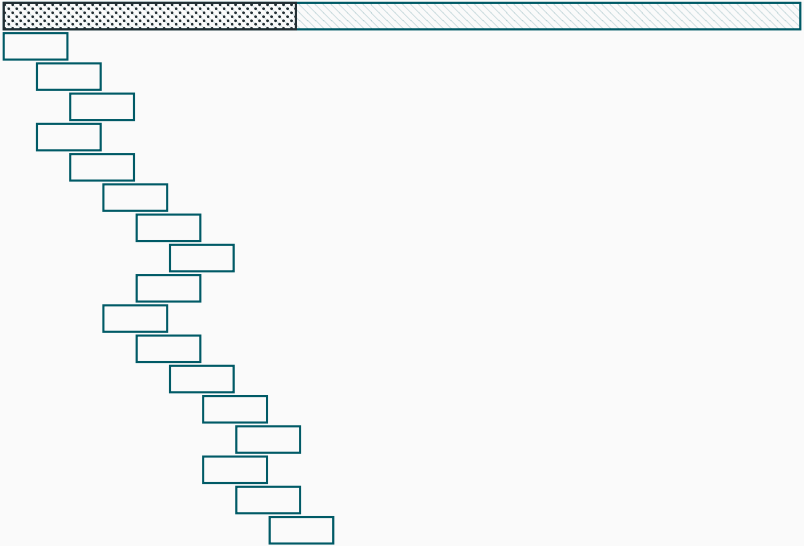

Quantitatively...

- Algorithmic tools: QR decomposition, Size-reduction
- Original analysis:

$$O\left(d^6 B^3\right)$$

- If very precautious one can use floating-point representation
- L^2 [Nguyen-Stehlé:2009]:

$$O\left(d^5 B(d+B)\right)$$

- [Neumaier-Stehlé:2016] (recursive strategy):

$$O\left(d^{4+\epsilon} B^{1+\epsilon}\right)$$

Generalization: towards algebraic lattices

Number fields and algebraic lattices

Number field

- Finite extension of $\mathbf{Q}$:

$$L \cong {}^{\mathbf{Q}[X]}\!/\!_{(P)}$$

- Ring of integers:

$$\mathcal{O}_L = \{\alpha \mid \exists R \in \mathbf{Z}[X] \text{ monic }, R(\alpha) = 0\}$$

Number fields and algebraic lattices

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- Finite extension of $\mathbf{Q}$:

$$L \cong {}^{\mathbf{Q}[X]}\!\diagup_{(P)}$$

- Ring of integers:

$$\mathcal{O}_L = \{\alpha \mid \exists R \in \mathbf{Z}[X] \text{ monic }, R(\alpha) = 0\}$$

For instance:

$$\mathcal{O}_{\mathbf{Q}} = \mathbf{Z}$$
$$\mathcal{O}_{\mathbf{Q}(i)} = \mathbf{Z}[i] = \{a + ib \mid a, b \in \mathbf{Z}\}$$
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Lattice

A (Euclidean) lattice Λ is a *discrete* subgroup of a Euclidean space (say $\mathbf{R}^n$).

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A (Euclidean) lattice Λ is a free $\mathbf{Z}$-module of finite rank, endowed with an inner product on $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$.

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Natural Hermitian structure

- $[L:\mathbf{Q}]$ embeddings $L \to \mathbf{C}$
- Archimedean embedding Σ:

$$\begin{array}{ccc} L \otimes_{\mathbf{Q}} \mathbf{R} & \to & \mathbf{R}^r \times \mathbf{C}^c \\ x & \mapsto & (\sigma(x))_{\sigma : L \to \mathbf{C}} \end{array}$$

- *Transport* the Hermitian structure to $L \otimes \mathbf{R}$:

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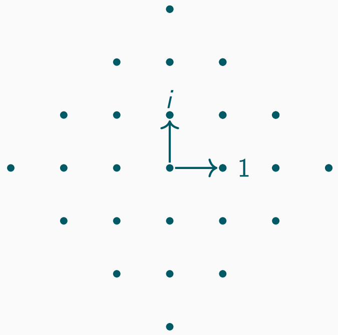

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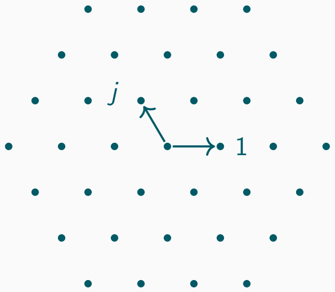

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Reduction of algebraic lattices

Techniques for the reduction of algebraic lattices

Techniques for the reduction of algebraic lattices

Module-SVP to Module-SVP type

QR-decomposition

Need an oracle...

Techniques for the reduction of algebraic lattices

Rank recursivity

Controlled precision

Parallel

fast-LLL

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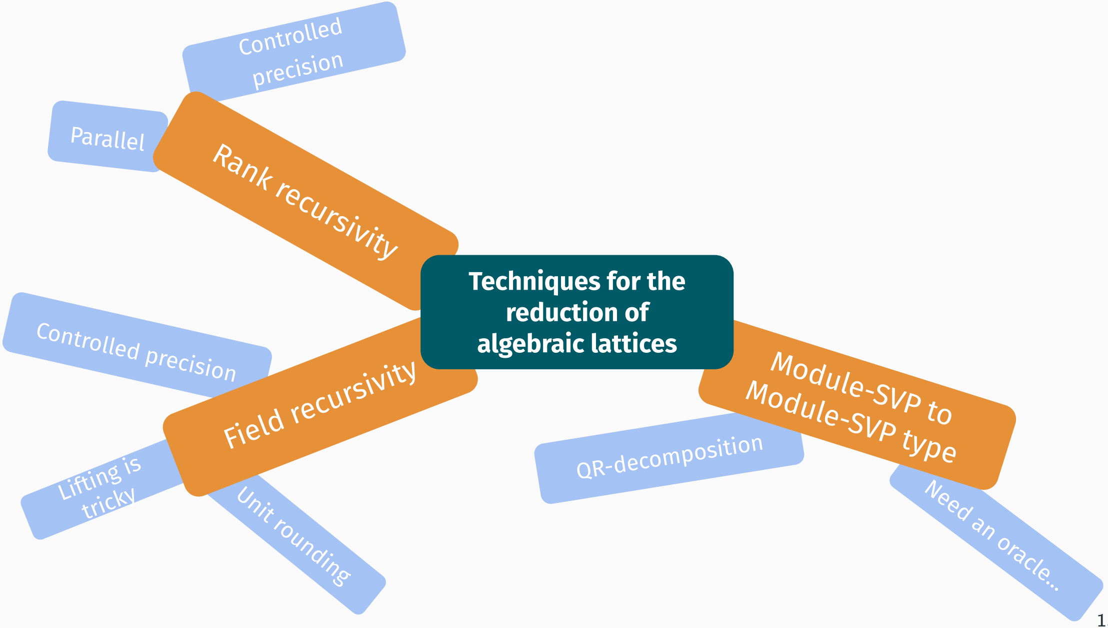
Techniques for the reduction of algebraic lattices
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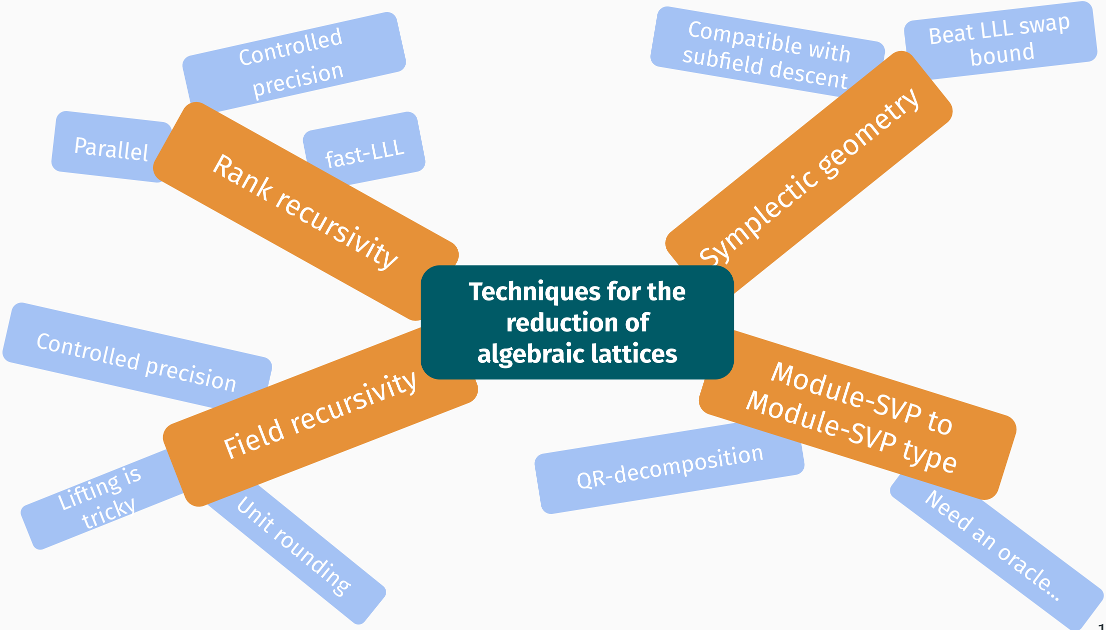
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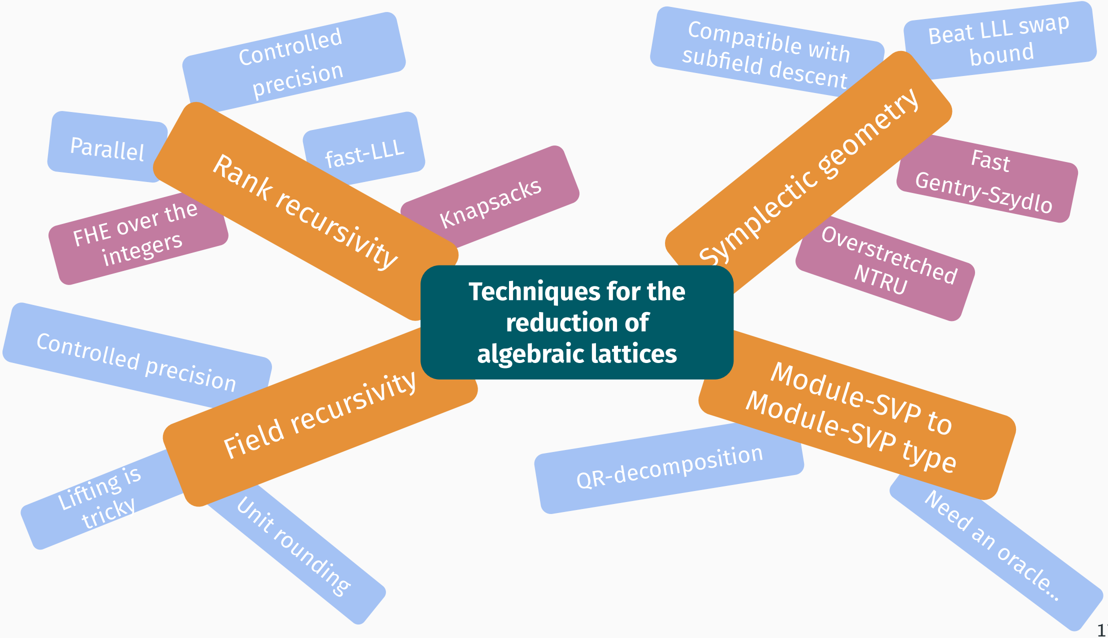
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Base reduction

Enhancing reduction over Z

$\rightarrow$ **Idea**: Use parallelisation and recursion on the rank.

Enhancing reduction over Z

Orthogonalize (M = QR)

for $j = 1$ **to** d **do**
  $Q_j \leftarrow M_j - \sum_{i=1}^{j-1} \frac{\langle M_j, Q_i \rangle}{\langle Q_i, Q_i \rangle} Q_i$
end for
return $R = \left(\frac{\langle Q_i, M_j \rangle}{\| Q_i \|} \right)_{1 \leqslant i < j \leqslant d}$;

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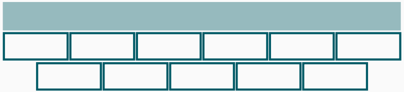

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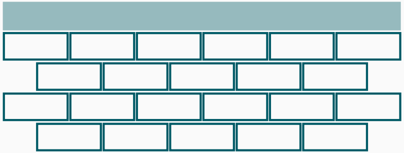

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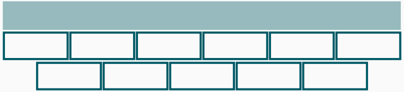
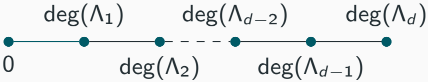


A round of local reductions acts as a *discretized* Laplacian operator on the profile "space":



- Reminiscent of the *diffusion property* of the solution of the heat equation

$$\frac{\partial u}{\partial t} = \alpha \Delta u$$

- *Characteristic time* is **quadratic in the diameter** of the space.

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On the complexity of the reduction

Complexity [E-Kirchner-Fouque 2019]

Let A be a matrix of dimension d with entries in $\mathbf{Z}$, with $\kappa(A) \leq 2^B$ such that $B \geq d$. Our reduction algorithm finds an integer vector x with

$$\|Ax\| \leq 2^{d/2} |\det A|^{1/d}.$$

Further, the *heuristic* running time is

$$\mathrm{O}\left(\frac{d^\omega}{(\omega-2)^2} \cdot \frac{B}{\log B} + d^2 B \log B\right).$$

Playing with number fields

$$\begin{tikzcd}
\mathsf{L} & \mathcal{O}_{\mathsf{L}} \arrow[l, hook'] \\
\mathsf{K} \arrow[u, hook] & \mathcal{O}_{\mathsf{K}} \arrow[l, hook'] \arrow[u, hook] \\
\mathsf{Q} \arrow[u, hook] & \mathsf{Z} \arrow[l, hook'] \arrow[u, hook]
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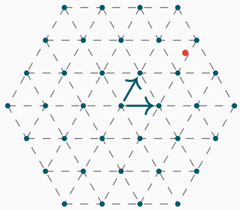
- Translated over $\mathcal{O}_K$: find the closest element in this ring: instance of CVP



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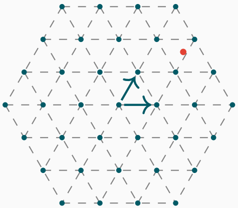

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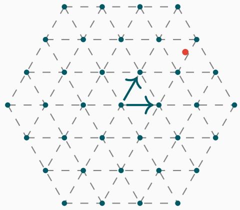

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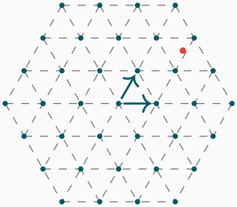

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  Hack: Use **units** to decrease the condition number and *lower the precision*.

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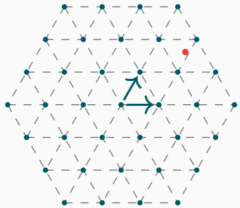

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\mathbf{Z} \arrow[u, hook] & M_{\mathbf{Z}} = \beta_1\mathbf{Z} \oplus \cdots \oplus \beta_k\mathbf{Z} & &
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\mathbf{Z} \arrow[u, hook] & M_{\mathbf{Z}} = \beta_1\mathbf{Z} \oplus \cdots \oplus \beta_k\mathbf{Z} \arrow[r, dash, dashed, "\text{reduce rank } 2" description] & \alpha_1\mathbf{Z} \oplus \alpha_2\mathbf{Z}
\end{tikzcd}
$$

General recursive strategy

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\begin{tikzcd}
\mathcal{O}_{K_h} & m_1\mathcal{O}_{K_h} \oplus \cdots \oplus m_d\mathcal{O}_{K_h} \arrow[r, dashed, "\text{reduce rk } 2"] & \alpha_1\mathcal{O}_{K_h} \oplus \alpha_2\mathcal{O}_{K_h} \arrow[dl, dashed, bend right, "\text{descend}"'] \\
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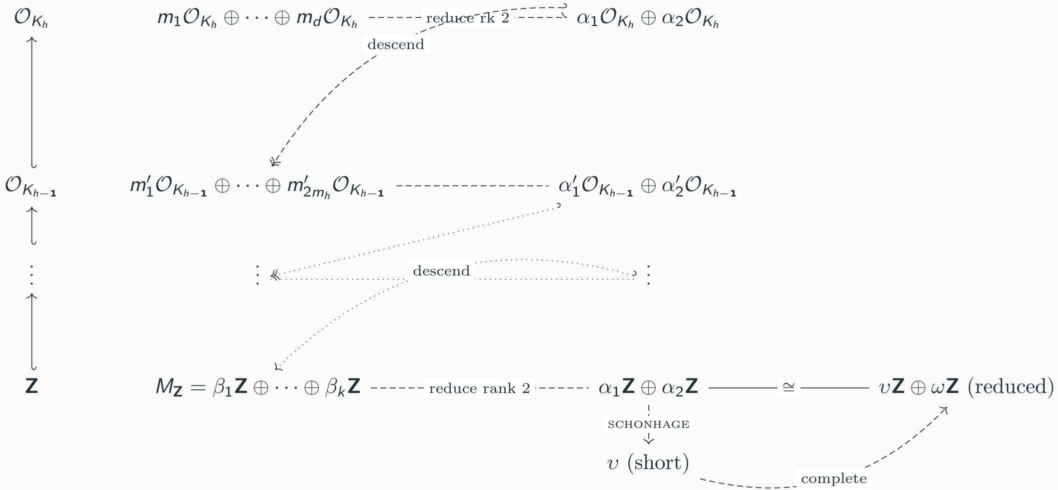
 & & v \text{ (short)}
\end{tikzcd}
$$

General recursive strategy



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\vdots \arrow[u, hook] & \vdots & \vdots \arrow[l, dotted, "\text{descend}"] \arrow[dl, dotted] & & \upsilon' \in M' \ \text{(short)} \\
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& & \upsilon \ \text{(short)} \arrow[urr, dashed, bend right, "\text{complete}"'] & &
\end{tikzcd}$$

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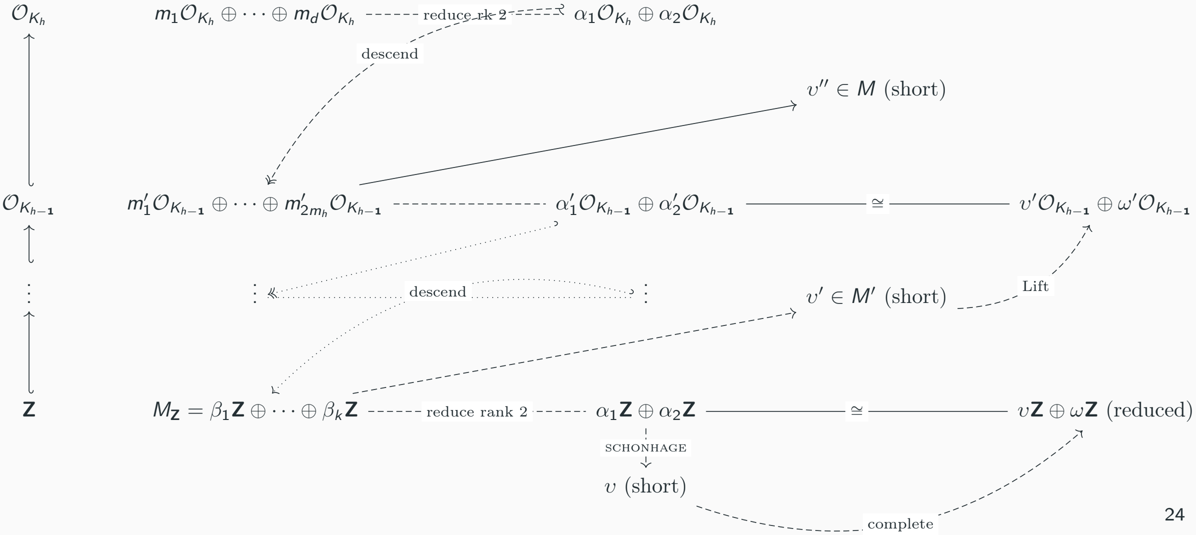

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$$

On the complexity of the reduction

Complexity [E-Kirchner-Fouque 2019]

Let f be a log-smooth integer. The complexity of the algorithm **Reduce** on rank two modules over $K = \mathbf{Q}[x]/\Phi_f(x)$, represented as a matrix M whose number of bits in the input coefficients is uniformly bounded by $B > n$, is *heuristically* a $\tilde{\mathrm{O}}(n^2 B)$ with $n = \varphi(f)$. The first column of the reduced matrix has its coefficients uniformly bounded by $2^{\tilde{\mathrm{O}}(n)} \operatorname{covol}(M)^{\frac{1}{2n}}$.

Faster with symplectic symmetries

$$\begin{array}{c}\mathrm{Sp}_{\omega}(2, \mathsf{L} \otimes \mathsf{R}) \\ \big\downarrow \\ \mathrm{Sp}_{\omega \cdot \tau}(2[\mathsf{L} : \mathsf{K}], \mathsf{K} \otimes \mathsf{R})\end{array}$$

A primer on symplectic geometry

A primer on symplectic geometry

Euclidean space	Symplectic space

A primer on symplectic geometry

Euclidean space

- Symmetric bilinear Form $\langle \cdot, \cdot \rangle$

Symplectic space

- Antisymmetric bilinear Form ω

A primer on symplectic geometry

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- Symmetric bilinear Form $\langle \cdot, \cdot \rangle$
- Transformation group: $O_n(\mathbf{R})$

Symplectic space

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A primer on symplectic geometry

Euclidean space

- Symmetric bilinear Form $\langle \cdot, \cdot \rangle$
- Transformation group: $O_n(\mathbf{R})$
- *Nice* bases: Orthonormal bases

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Symplectic space

- Antisymmetric bilinear Form ω
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$$\begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$$

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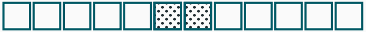

Going further with symplectic symmetries

$$\begin{tikzcd}
\mathbf{K}_h & \mathcal{O}_{\mathbf{K}_h} \arrow[l, hook'] \\
\mathbf{K}_{h-1} \arrow[u, hook, "r"] & \mathcal{O}_{\mathbf{K}_{h-1}} \arrow[l, hook'] \arrow[u, hook] \\
\vdots \arrow[u, hook] & \vdots \arrow[u, hook] \\
\mathbf{Q} \arrow[u, hook] & \mathbf{Z} \arrow[l, hook'] \arrow[u, hook]
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Going further with symplectic symmetries

J_h

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\mathbf{Q} \arrow[u, hook] & \mathbf{Z} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

$J_h \in \bigwedge^2(K_h^2)$ is the determinant form:

$$J_h\left(\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}\right) = x_0 y_1 - x_1 y_0$$

M is J_h-symplectic iff $\det M = 1$.

Going further with symplectic symmetries

$$
\begin{tikzcd}
J_h \arrow[d, dotted, "\tau"] & \mathbf{K}_h & \mathcal{O}_{\mathbf{K}_h} \arrow[l, hook'] \\
J_h^{(1)} & \mathbf{K}_{h-1} \arrow[u, hook, "r"] & \mathcal{O}_{\mathbf{K}_{h-1}} \arrow[l, hook'] \arrow[u, hook] \\
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Compatibility

Let M be a 2×2 matrix over K_h which is J_h-symplectic, then its descent $M' \in K_{h-1}^{2d_h \times 2d_h}$ is J_h'-symplectic.

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\vdots \arrow[d, hook] & \vdots \arrow[d, dashed] & \vdots \arrow[u] & \vdots \arrow[u] \\
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Improved complexity

Improved complexity [E-Kirchner-Fouque 2019]

Select an integer f a power of $q = \mathrm{O}(\log f)$ and let $n = \varphi(f)$. The complexity for reducing matrices M with condition number lower than 2^B, of dimension two over $L = \mathbf{Q}[x]/\Phi_f(x)$ with B the number of bits in the input coefficients is *heuristically*

$$\tilde{\mathrm{O}}\left(n^{2+\varepsilon(q)}B\right) + n^{\mathrm{O}(\log\log n)}, \qquad \varepsilon(q) = \frac{\log(1/2 + 1/2q)}{\log q} < 0$$

and the first column of the reduced matrix has coefficients bounded by

$$2^{\tilde{\mathrm{O}}(n)}\left|N_{K_h/\mathbf{Q}}(\det M)\right|^{\frac{1}{2n}}.$$

Open problems

- Getting further using higher-order symplectic structures

Exploitation of the symplectic symmetries:

1. **Decrease** the complexity, but...
2. **Increase** the approximation factor

Open problems

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Seek for transformations preserving arbitrary non-degenerate alternate forms (for example the *volume form*)

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- Non uniqueness of higher-order symplectic structures (no Darboux' structure theorem)
- Find a descent compatible with this additional structure

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- Getting further using higher-order symplectic structures

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Over $\mathcal{O}_L$ a projective module is of the shape: $\alpha_1\mathfrak{a}_1 \oplus \cdots \oplus \alpha_n\mathfrak{a}_n$

- Need to adapt the lifting to ideals [Cohen]

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- 2-elements representation: multiplying $\mathfrak{a} = \alpha_1 \mathcal{O}_L + \alpha_2 \mathcal{O}_L, \mathfrak{b} = \beta_1 \mathcal{O}_L + \beta_2 \mathcal{O}_L$ consists in the reduction of the ideal generated by $(\alpha_i \beta_j)_{1 \leqslant i,j \leqslant 2}$ (module spanned by 4 elements)

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Cross recursive algorithms: reduction and ideal multiplication

Thank you !