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- Reality: Perhaps more (1) and (4).
- Crucial help: from Continuous Mathematics, Developments in Randomized algorithms.

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- Non-negative Matrix Factorization (NMF)

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3. Hybrids also possible.
4. Here more (1) than (2). More useful for large matrices peculiar to a single context, like Web, FB graph, ...

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  3. Here, think of (1) generally.

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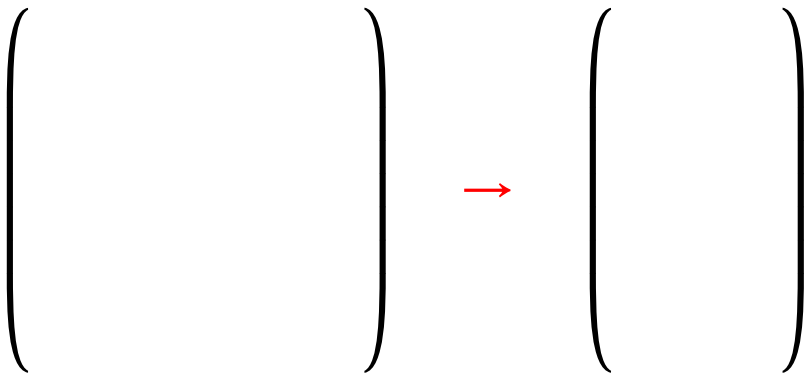
- 

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- Alternative Scheme: Draw a sample of entries, set others to zero. Sparsity gain rather than reduction in dimensions. Achlioptas, McSherry; Bhojanapalli, Jain and Sanghavi.

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  - Drineas, Kannan, Mahoney (Approx) Matrix Multiplication in $O^*(n^2)$ time.

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- Also, one can first do length squared sampling to pick s columns, then again do length squared sampling to pick s rows (to form a $s \times s$ matrix) and so use only $O(1)$ RAM space. Proof gets very complicated.

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- But still don't know for an unsampled row what linear combination it is. Suppose we also pick a sample of $100k$ columns. Now, intuitively, (if the rows are in general position), there should be a unique linear way of expressing each row (in the $100k$-column matrix) in the row space.

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- Question in this form arose in Volume computation and log-concave sampling Kannan, Lovász, Simonovits.

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- Rudelson, Vershynin Theorem implies for log-concave probability densities on $\mathbf{R}^d$, $O^*(d)$ samples suffice (improving earlier answers of Pisier and Bourgain).

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- Not good enough. But if only make A_G an isometry, $||A_G||_2|\mathbf{v}|^2 = |\mathbf{v}^T A_G|^2$ and take $s \geq cn$, it all works.

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- All of the above true for **any matrix** A.
- Computing p_j involves finding W. Spielman, Srivatsava For node-edge adjacency matrix of a graph, can be done in linear time.
- Open question: Are there other class of interesting matrices for which p_j can be computed fast?

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- An alternative way to get the same theorem: Draw a sample of $r =$poly(k/ε) sample columns of A with probabilities proportional to the square of the volume of the simplex spanned by them. Deshpande, Rademacher and Vempala Volume Sampling, Determinental process

JAGUAR

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- Better results known under assumptions on the tensor known used: (for eg. orthogonal rank 1 decomposition) - Anandkumar, Foster, Hsu, Kakade, Liu

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