Quantum attacks on CSIDH: an overview

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University of Bristol

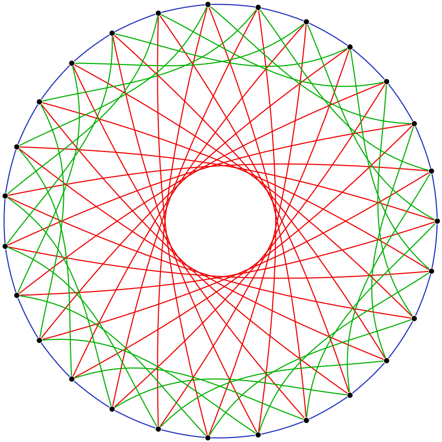
Based on joint work with
Daniel J. Bernstein, Tanja Lange, and Lorenz Panny

quantum.isogeny.org

Why CSIDH?

- Drop-in post-quantum replacement for (EC)DH
- Non-interactive key exchange (full public-key validation); previously an open problem post-quantumly
- Smallest keys of all post-quantum key exchange candidates
- Competitive speed: 50-60ms for a full key exchange

CSIDH: a picture



Secret key: path on the graph
Public key: end points of path.

Quantum complexity analysis

Recall Kuperberg's algorithm from David Jao's talk.

2011 Kuperberg estimates time complexity $2^{(\sqrt{2}+o(1))\sqrt{\log_2 p}}$, improvement on 2003 Kuperberg: $2^{(1.77+o(1))\sqrt{\log_2 p}}$.

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- Can the power of $\log_2 p$ be reduced?
- If not, can the constant $\sqrt{2}$ be improved? (Last improvement: 2011).
- If not, what's the smallest o(1)? Important for proposing parameters! (See next talk).

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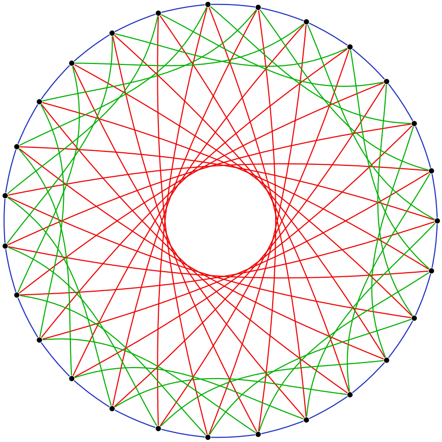
One CSIDH query: isogenies



Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
Edges: 3-, 5-, and 7-isogenies.

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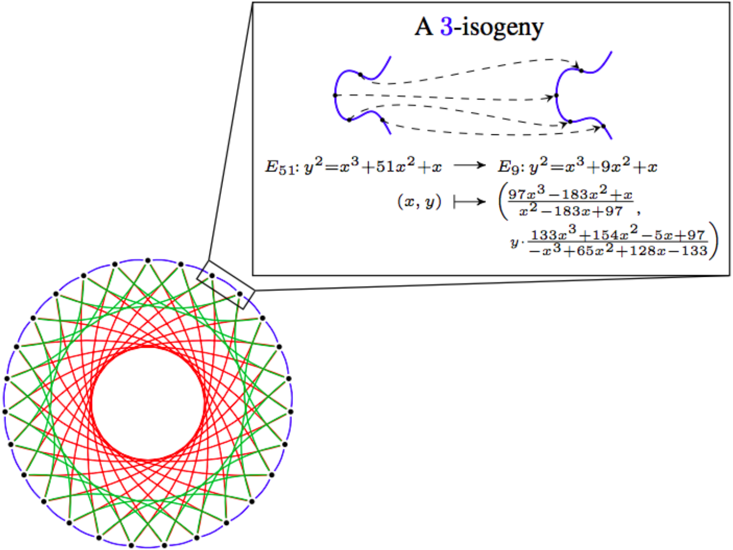
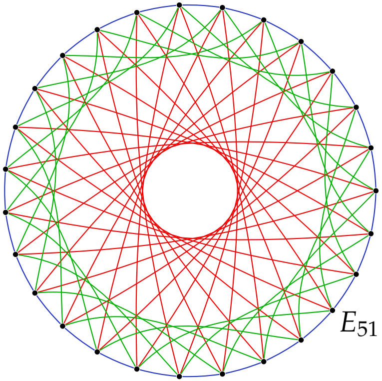


Computing isogenies

Aim: given curve E_A, find a neighbour in the isogeny graph



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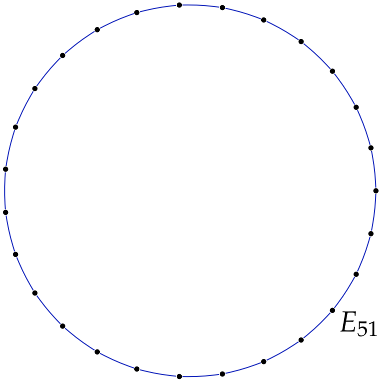

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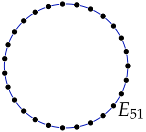

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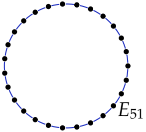

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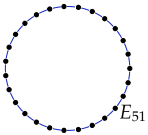

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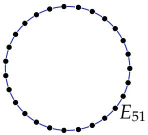

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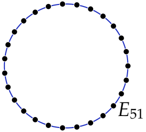

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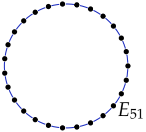

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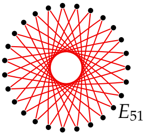

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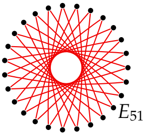

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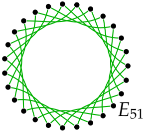

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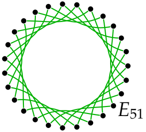

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- Choose a random $\mathbb{F}_{419}$-point $P = (x, y)$ on E_{51}
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- With probability $\frac{6}{7}$, $60 \cdot P$ has order 7
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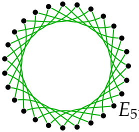
Computing isogenies

Aim: given curve E_A, find a neighbour in the ℓ-isogeny graph



- Recall: $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$
- Choose a random $\mathbb{F}_p$-point $P = (x, y)$ on E_A
- P has order dividing $p + 1$.
- With probability $\frac{\ell-1}{\ell}$, $\frac{p+1}{\ell} \cdot P$ has order ℓ.*
- Find map with kernel = $\langle \frac{p+1}{\ell} \cdot P \rangle$
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[BLMP] Gives many optimizations / more complex variants–trying to mitigate these problems.

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Open question:
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- Number of queries: see next talk.

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- Understanding the error tolerance of Kuperberg's algorithm is essential to obtain accurate concrete numbers.
- Advances in quantum error correction would also massively change the complexity.

Open questions: summary

- How do oracle errors interact with Kuperberg's algorithm?
- What kind of overheads come from handling large numbers of qubits?
- Is there a quantum algorithm that does better than L(1/2)?
- Can we decrease the cost of one query?

Open questions: summary

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Thank you!

References

BLMP Bernstein, Lange, Martindale, and Panny,
Quantum circuits for the CSIDH: optimizing quantum evaluation of isogenies,
Eurocrypt 2019, `quantum.isogeny.org`.

CLMPR Castryck, Lange, Martindale, Panny, and Renes,
CSIDH: An efficient post-quantum commutative group action,
Asiacrypt 2018, `csidh.isogeny.org`.

Credits to my coauthors Daniel J. Bernstein, Tanja Lange, and Lorenz Panny for many of the contents of this presentation.