Overview of Quantum Algorithmic Tools

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Quantum Cryptanalysis of Post-Quantum Cryptography
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Block-encodings and

Quantum Singular Value Transformation

Block-encoding

A way to represent large matrices on a quantum computer efficiently

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Implementing arithmetic operations on block-encoded matrices

- Given block-encodings A_j we can implement convex combinations.
- Given block-encodings A, B we can implement block-encoding of AB.

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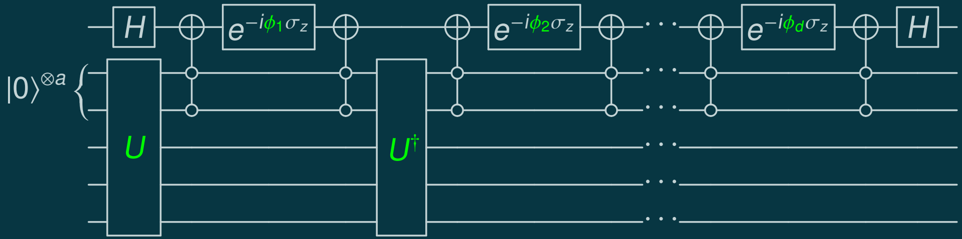

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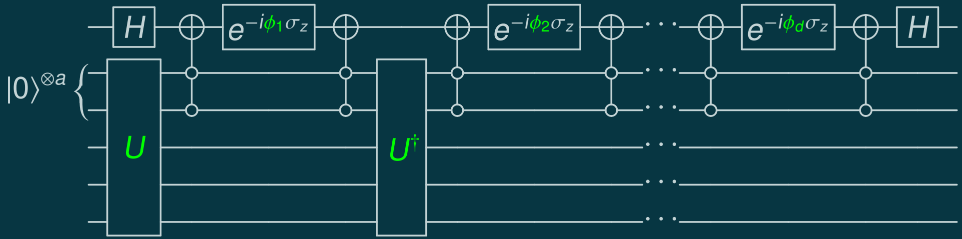

Simmilar result holds for even polynomials.

Amplitude amplification and estimation

Fixed-point amplitude ampl. (Yoder, Low, Chuang 2014)

Amplitude amplification problem: Given U such that

$$U|\bar{0}\rangle = \sqrt{p}|0\rangle|\psi_{\text{good}}\rangle + \sqrt{1-p}|1\rangle|\psi_{\text{bad}}\rangle, \qquad \text{prepare } |\psi_{\text{good}}\rangle.$$

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Note that $(|0\rangle\langle 0| \otimes I)U(|\bar{0}\rangle\langle\bar{0}|) = \sqrt{p}|0, \psi_{\text{good}}\rangle\langle\bar{0}|$; we can apply QSVT.

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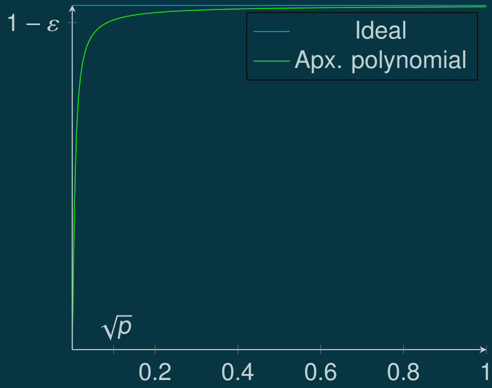
Amplification using QSVT (degree $\approx \log(1/\varepsilon)/\sqrt{p}$)



Detecting a bias in a quantum sampler

Suppose we are given U such that

$U|\bar{0}\rangle = \sqrt{p'}|0\rangle|\psi_{\text{good}}\rangle + \sqrt{1-p'}|1\rangle|\psi_{\text{bad}}\rangle$, distinguish $p' \leq p - \delta$ vs. $p + \delta \leq p'$.

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Can be solved by QSVT (or using amplitude estimation).

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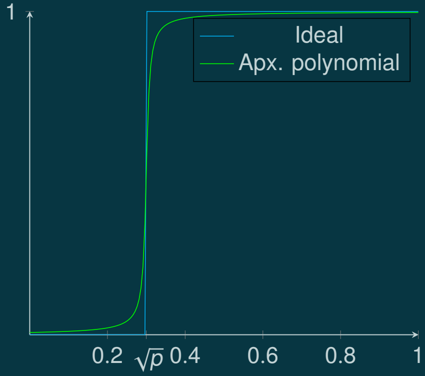
Bias detection using QSVT (degree $\approx \log(1/\varepsilon)/\delta$)



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Suppose we have a sampling algorithm sampling from a random variable X.

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We can estimate $\mathbb{E}[X]$ with ε precision using $\approx \frac{\sigma^2}{\varepsilon^2}$ samples.

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- Implement sampler as a quantum circuit
- Replace random input seed with Hadamard gates
- Apply amplitude estimation + combine with other tricks

Quantum walks

Discrete-time random / quantum walks

Discrete-time random walk on a weighted graph

Let $G = (V, E)$ be a finite simple graph, with non-negative edge-weights $w\colon E \to \mathbb{R}_+$.
Transition probability in one step (stochastic matrix)

$$P_{vu} = \Pr(\text{step to } v \,|\, \text{being at } u) = \frac{w_{vu}}{\sum_{v' \in U} w_{v'u}}$$

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Basic primitives – classical

$$\begin{aligned} \text{(Setup) } S &: \text{ sample } v \text{ with probability } \sigma_v \\ \text{(Update) } U &: \text{ given } u \text{ sample } v \text{ with probability } P_{vu} \\ \text{(Check) } C &: \text{ given } v \text{ check if it is marked, i.e., } v \in M? \end{aligned}$$

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$$\text{(Walk operator) } W\colon U^\dagger \cdot \mathrm{SWAP} \cdot U = \begin{bmatrix} P & \cdot \\ \cdot & \cdot \end{bmatrix}$$

High-level explanation of quadratic speed-ups

Quantum fast-forwarding (Apers & Sarlette 2018)

We can implement a unitary V such that

$$(\langle 0| \otimes I) V (|0\rangle \otimes I) \overset{\varepsilon}{\approx} P^t$$

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Proof:

$$x^t \approx \sum_{k-\sqrt{t}}^{\sqrt{t}} \binom{2t}{t+k} T_k(x)$$

Szegedy quantum walk based search

Suppose we have some unknown marked vertices $M \subset V$.

Quadratically faster hitting

Hitting time: expected time to hit a marked vertex starting from the stationary distr.
Starting from the quantum state $\sum_{v\in V} \sqrt{\pi_v}|v\rangle$ we can

- detect the presence of marked vertices ($M \neq 0$) in time $\mathcal{O}\left(\sqrt{HT}\right)$ (Szegedy 2004)
- find a marked vertex in time $\mathcal{O}\left(\frac{1}{\sqrt{\delta\varepsilon}}\right)$ (Magniez, Nayak, Roland, Sántha 2006)
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Starting from arbitrary distributions

Starting from distribution σ on some vertices we can

- detect marked vertices in square-root commute time $\mathcal{O}\left(\sqrt{C_{\sigma,M}}\right)$ (Belovs 2013)
- find a marked vertex in time $\widetilde{\mathcal{O}}\left(\sqrt{C_{\sigma,M}}\right)$ (Piddock; Apers, **G**, Jeffery 2019)

Walks on the Johnson graph (Sántha `arXiv:0808.0059`)

Vertices: $\{S \subset N \colon |S| = K\}$; Edges: $\{(S, S') \colon |S \triangle S'| = 2\}$

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- Black box: Computes f on inputs corresponding to elements of $[n]$
- Question: Are there any $i \neq j \in [n] \times [n]$ such that $f(i) = f(j)$?
- Query complexity: $\mathcal{O}(n^{2/3})$ (Ambainis 2003) $\Omega(n^{2/3})$ (Aaronson & Shi 2001)

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Matrix Product Verification

- Black box: Tells any entry of the $n \times n$ matrices A, B or C.
- Question: Does $AB = C$ hold?
- Query complexity: $\mathcal{O}(n^{5/3})$ (Buhrman, Špalek 2004)

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Are there k distinct elements mapped to the same image?

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Quantum time-space trade-offs?

- Hamoudi & Magniez (arXiv: yesterday) Progress towards conjectured

$$T^2 S \geq \widetilde{\Omega}(n^2)$$

Quantum linear equation solving (HHL)

Direct implementation of the pseudoinverse (HHL)

$Ax = b$: solve the regression problem in a quantum sense – output $|x\rangle \propto A^+|b\rangle$

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Singular value decomposition and pseudoinverse

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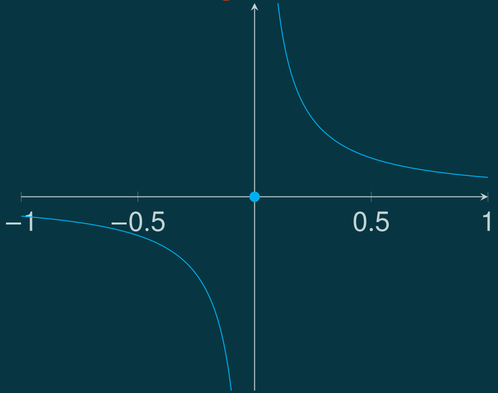
Implementing the pseudoinverse using QSVT



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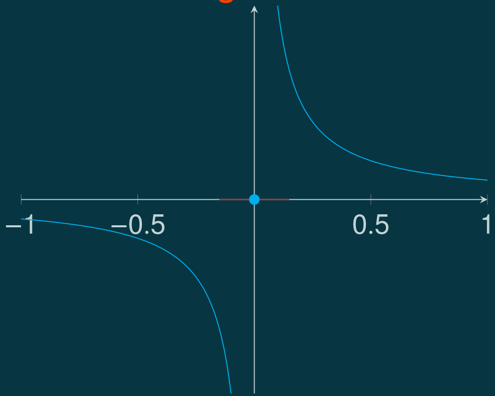

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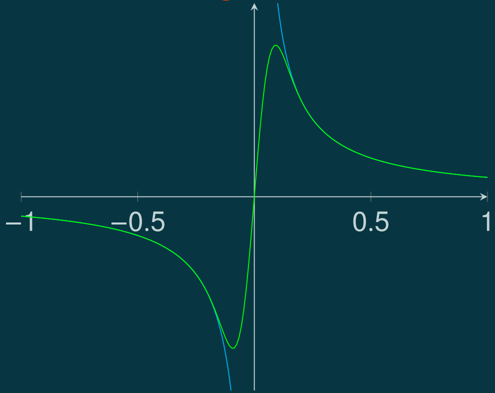

Direct implementation of the pseudoinverse (HHL)

$Ax = b$: solve the regression problem in a quantum sense – output $|x\rangle \propto A^+|b\rangle$

Singular value decomposition and pseudoinverse

Suppose $A = \sum_i \varsigma_i |w_i\rangle\langle v_i| \in \mathbb{C}^{n\times m}$ is a singular value decomposition.
Then the pseudoinverse of A is $A^+ = \sum_i 1/\varsigma_i |v_i\rangle\langle w_i|$ (note $A^\dagger = \sum_i \varsigma_i |v_i\rangle\langle w_i|$).

Implementation cost (s = sparsity, κ =condition number, ε = precision)

$$O(T \cdot s \cdot \kappa \cdot \mathrm{polylog}(nm\kappa/\varepsilon)),$$

assuming in time T we can

- prepare $|b\rangle$,
- find and compute non-zero elements in a row / column of A.

Potentially exponential speed-ups?

Application for Boolean equations – Chen & Gao 2017

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Convert to a complex linear equation system (rough sketch)

- Set $D > \max_i \deg(f_i)$, and denote by m_D be the set of degree $\leq D$ monomials

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Bad news: $\kappa^2\geq\|x\|^2/\|b\|^2\geq\frac{\#\{\text{non-zero variables}\}}{c}=\frac{\binom{k+D-1}{k-1}}{c}\approx\left(\frac{eD}{k}\right)^k$

Classical brute-force: $\sum_{j=0}^{k}\binom{n}{k}\approx\left(\frac{en}{k}\right)^k\implies$ Does not seem to be useful if $D\geq n$

Summary of some relevant quantum speed-ups

Optimization

In general we want to find the best solution $\min_{x \in X} f(x)$

- Unstructured: can be solved with $\mathcal{O}\left(\sqrt{|X|}\right)$ queries (Dürr & Høyer 1996)

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Discrete structures:

- Finding the shortest path in a graph
  $\mathcal{O}\left(n^2\right)$ (Dijkstra 1956); quantum $\widetilde{\mathcal{O}}\left(n^{3/2}\right)$ (Dürr, Heiligman, Høyer, Mhalla 2004)
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- NP–hard problems:
  Quadratic speed-ups for Schöning's algorithm for 3-SAT (Ampl. ampl.)
  Quadratic speed-ups for backtracking (Montanaro '15, Ambainis & Kokainis '17)
  Polynomial speed-ups for dynamical programming, e.g., TSP $2^n \to 1.73^n$
  (Ambainis, Balodis, Iraids, Kokainis, Prūsis, Vihrovs 2018)

Continuous optimization

Convex optimization

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- Polynomial speed-up for estimating volumes of convex bodies
  (Chakrabarti, Childs, Hung, Li, Wang, Wu 2019)

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- Estimating the histogram to ε-precision
  Query and time complexity $\widetilde{\mathcal{O}}(1/\varepsilon)$ (Apeldoorn 2020)