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Faster quantum and classical SDP approximations for quadratic binary optimization

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Motivatio

The problem

/leta-algorithm

feasibility

iii. Hamiltonian Updates

untime analysis

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Quantum algorithms for optimization

- quantum algorithms for optimization tasks is a promising "new" area
- mild speedups, but many important applications many applications
- important example: semidefinite programming (SDPs)
- existing quantum algorithms don't always yield improvements
- "open" challenge: relaxations of binary quadratic problems

ideas

- bundle many linear constraints together (convex constraints)
- develop primal only classical algorithm (mirror descent)
- embed quantum simulation as fast subroutine (Gibbs sampling)

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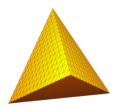
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Binary quadratic optimization

captures many important problems:

- i MAXCUT and CUTNORM
- ii community detection
- iii semi-discrete matrix factorization
- iv Ising model and spin glasses



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SDP relaxation

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 $\underset{\boldsymbol{x} \in \mathbb{R}^n}{\operatorname{maximize}} \quad \operatorname{tr}(\boldsymbol{A} \boldsymbol{x} \boldsymbol{x}^*)$ subject to $\mathbf{x} \in \{\pm 1\}^n$

 $\mathrm{tr}\left(\boldsymbol{A}\;\boldsymbol{X}\right)$ maximize $X \in \mathbb{S}^n$

subject to $\operatorname{diag}(X) = 1$ $X \succ 0$

rank(X) = 1

convex relaxation:

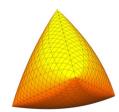
$$f(\boldsymbol{X}) = \operatorname{tr}(\boldsymbol{A} \ \boldsymbol{X})$$
 is linear

 $X \in \mathcal{C}_1 \cap \mathcal{C}_2$ where

 $C_1 = \{ \boldsymbol{X} : \operatorname{diag}(\boldsymbol{X}) = 1 \}$ affine subspace

 $C_2 = \{ \boldsymbol{X} : \boldsymbol{X} \succ \boldsymbol{0} \}$ convex cone

actually a **SDP**, but tr(X) = n



Fundamental problem for this talk

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 $\underset{\boldsymbol{X} \in \mathbb{S}^n}{\text{maximize}} \quad \operatorname{tr}\left(\frac{1}{\|\boldsymbol{A}\|}\boldsymbol{A}\;\boldsymbol{X}\right)$ subject to $\operatorname{diag}(\boldsymbol{X}) = \frac{1}{2}\mathbf{1}$

 $\operatorname{tr}(\boldsymbol{X}) = 1, \ \boldsymbol{X} \succ \boldsymbol{0}$

 $(X \in C_1)$ $(X \in S)$

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phase I: optimization problem ⇒ feasibility problem
 phase II: develop quantum-inspired meta-algorithm
 quantum boost: use quantum subroutines

inspiration: matrix multiplicative weights, mirror descent

Optimization \Rightarrow **feasibility**

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objective function $f(\boldsymbol{X})$ is *linear* and *bounded* instead of optimizing $f(\boldsymbol{X})$ directly, choose $\lambda \in [-1,1]$ and ask: is there a feasible \boldsymbol{X} that obeys $f(\boldsymbol{X}) \leq \lambda$?

Binary search

 $\mathcal{O}(2\log(1/\epsilon)) = \tilde{\mathcal{O}}(1)$ questions (with varying λ) nail down $f(\textbf{\textit{X}}_\sharp) \pm \epsilon$

Reformulate feasibility problem

• S_n is the set of all density matrices

Quantum-inspired change of variables

 A_λ is half-space • \mathcal{D}_n is affine subspace Caltech

task: for
$$\ddot{\pmb{A}} = \frac{1}{\|\pmb{A}\|} \pmb{A}$$
 and $\lambda \in [-1,1]$ solve

find
$$X \in \mathbb{S}^n$$

subject to $\operatorname{tr}\left(\tilde{\boldsymbol{A}}\boldsymbol{X}\right) \leq \lambda$

 $\operatorname{diag}(\boldsymbol{X}) = \frac{1}{\pi}\boldsymbol{I}$

 $X = \rho_H = \frac{\exp(-H)}{\operatorname{tr}(\exp(-H))} \in \mathcal{S}_n$ (Gibbs state)

 $(X \in S_n)$

 $\operatorname{tr}(\boldsymbol{X}) = 1. \ \boldsymbol{X} \succ \boldsymbol{0}$

 $(X \in \mathcal{D}_n)$

 $(\boldsymbol{X} \in \mathcal{A}_{\lambda})$

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Hamiltonian Updates

 $m{X}\mapsto m{
ho_H} = rac{\exp(-m{H})}{\operatorname{tr}(\exp(-m{H})}$ automatically ensures $m{X}\in\mathcal{S}_n$

find
$$\boldsymbol{H} \in \mathbb{S}^n$$

subject to
$$\operatorname{tr}(\tilde{\boldsymbol{A}} \boldsymbol{\rho_H}) \leq \lambda$$

$$\operatorname{diag}(\boldsymbol{\rho_H}) = \frac{1}{n}\boldsymbol{I}$$

$$(
ho_{m{H}}\in\mathcal{A}_{\lambda})$$

$$(
ho_{m{H}}\in\mathcal{D}_n)$$

Hamiltonian Updates:

- **1** start with H = 0 ("infinite temperature")
- 2) check if $\rho_{H} \in \mathcal{A}_{\lambda}$ and $\rho_{H} \in \mathcal{D}_{n}$

if true we are done else update **H** to penalize infeasible directions^a

3 loop (at most) T times

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^afind separating hyperplane **P** and update $\mathbf{H} \leftarrow \mathbf{H} + \epsilon \mathbf{P}$



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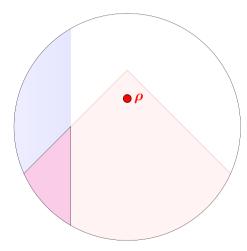
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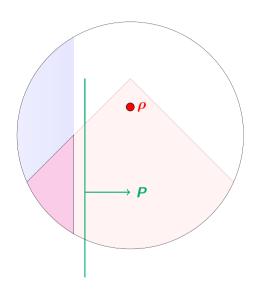
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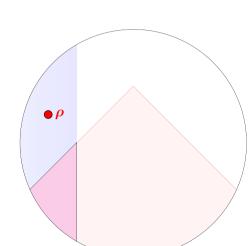
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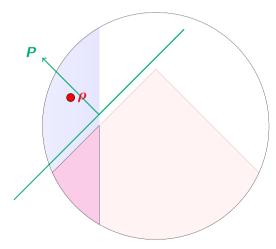
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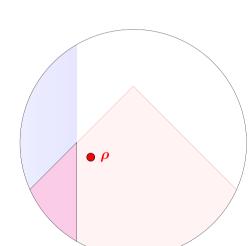
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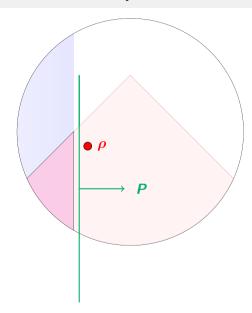
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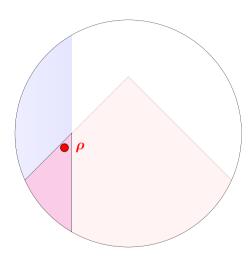
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Hamiltonian Updates: convergence

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Theorem (Brandão, RiK, França)

Hamiltonian Updates finds an approximately feasible point after (at most) $T = \lceil 16 \log(n)/\epsilon^2 \rceil + 1 = \tilde{\mathcal{O}}(1)$ steps. Otherwise, the problem is infeasible.

proof idea:

- relative entropy between $ho_0 = \frac{1}{n}I$ and any feasible point ho^* is $\leq \log(n)$
- show that each iteration makes constant progress in relative entropy:

$$S(oldsymbol{
ho}^* \| oldsymbol{
ho}_{t+1}) - S(oldsymbol{
ho}^* \| oldsymbol{
ho}_t) \leq -rac{\epsilon^2}{16}$$

 \Rightarrow convergence after (at most) T steps, or $S(
ho^* \|
ho_T) < 0$

optimization context: mirror descent with von-Neumann entropy potential

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Hamiltonian Updates: computational cost

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- Hamiltonian Updates solves feasibility problem in $\mathcal{O}(\log(n)/\epsilon^2) = \tilde{\mathcal{O}}(1)$ steps
- each step requires three subroutines:
 - (i) compute $ho_{m{H}} = rac{\exp(-m{H})}{\operatorname{tr}(\exp(-m{H}))}$
 - (ii) $ho_{m{H}} \in \mathcal{A}_{\lambda}$: check $\operatorname{tr}(\tilde{m{A}} \; m{
 ho_{m{H}}}) \leq \lambda$; output $m{P} = \tilde{m{A}}$
 - (iii) $\rho_H \in \mathcal{D}_n$: check diag $(\rho_H) = \frac{1}{n}\mathbf{1}$; output $P = \sum_i \mathbb{I}\left\{\langle \mathbf{e}_i, \rho_H \mathbf{e}_i \rangle > \frac{1}{n}\right\} \mathbf{e}_i \mathbf{e}_i^t$
- naive cost:
 - (i) $\mathcal{O}(n^3)$
 - (ii) $\mathcal{O}(ns)$ $s = (row) sparsity(\tilde{\mathbf{A}})$
 - (iii) $\mathcal{O}(n)$
- naive total cost: $\tilde{\mathcal{O}}(n^4s)$ (not very impressive yet)

Hamiltonian Updates: classical implementation

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fact: Hamiltonian updates is designed to be robust

 \Rightarrow implementing subroutines up to accuracy ϵ still yields an approximately feasible solution (and correctly flags infeasibility)

classical boost:
$$\exp(-\boldsymbol{H}) \simeq \sum_{k=0}^{\ell} \frac{\boldsymbol{H}^k}{k!}$$
, $\ell = \mathcal{O}(\log(n)/\epsilon) = \tilde{\mathcal{O}}(1)$

Theorem (Brandão, RiK, França; 2019)

Hamiltonian Updates approximately solves quadratic SDP relaxations in classical runtime $\mathcal{O}\left(n^2s\log(n)/\epsilon^{12}\right) = \tilde{\mathcal{O}}(n^2s)$, where $s = (row)sparsity(\mathbf{A})$.

$$\begin{array}{ll} \text{maximize} & \operatorname{tr}\left(\frac{1}{\|\boldsymbol{A}\|}\boldsymbol{A}\;\boldsymbol{X}\right) \\ \text{subject to} & \operatorname{diag}(\boldsymbol{X}) = \boldsymbol{1} \\ & \boldsymbol{X} \succ \boldsymbol{0}. \end{array}$$

- 1 best existing general algorithm: $\tilde{\mathcal{O}}(n^{2.5}s)$
- 2 approx. discrepancy: $\epsilon n \|\mathbf{A}\|$ vs. $\epsilon \|\mathbf{A}\|_{\ell_1}$
- 3 favorable for generic problem instances
- $oldsymbol{4}$ no speedup for MAXCUT

Hamiltonian updates: quantum implementation

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classical bottleneck: compute Gibbs states $ho_H = \frac{\exp(-H)}{\operatorname{tr}(\exp(-H))}$ quantum speedup:

prepare copies of ρ_H on quantum computer $\tilde{\mathcal{O}}(\sqrt{ns}s^{o(1)})$ estimate $\operatorname{tr}(\tilde{\mathbf{A}}\;\rho_H)$ via phase estimation $\mathcal{O}(1/\epsilon^2)$ copies estimate $\operatorname{diag}(\rho_H)$ via computational basis measurements $\mathcal{O}(n/\epsilon^2)$ copies

Theorem (Brandão, RiK, França; 2019)

Hamiltonian Updates approximately solves binary quadratic SDP relaxations in quantum runtime $\tilde{\mathcal{O}}(n^{1.5}(\sqrt{s})^{1+o(1)})$.

- 1 first quantum speedup for combinatorial SDP relaxation
- ② beats classical runtimes $\tilde{\mathcal{O}}(n^2s)$ and $\tilde{\mathcal{O}}(n^{2.5}s)$
- 3 classical access to (approx.) optimal Hamiltonian \Rightarrow data processing

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Details about quantum subroutine

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important design feature: Hamiltonians are very structured:

$$\mathbf{H} = \alpha \tilde{\mathbf{A}} + \beta \mathbf{D}, \ \alpha, \beta = \mathcal{O}(\log(n)/\epsilon)$$

- 1 use [Poulin, Wojcan; 2009] to reduce task of preparing ρ_H to simulating time evolution $(\mathcal{O}(\sqrt{n}))$ invocations
- 2 use [Childs, Wiebe; 2012] to split up time evolution (negligible overhead)
- 3 [Low; 2019]: implementing $\exp(it\alpha\tilde{\mathbf{A}})$ costs $\tilde{\mathcal{O}}(\sqrt{s}^{1+o(1)})$
- 4 [Prakash; 2014] implementing $\exp(it\beta \mathbf{D})$ with quantum RAM costs $\tilde{\mathcal{O}}(n)$
- \Rightarrow total cost: $\tilde{\mathcal{O}}(n^{1.5}\sqrt{s}^{1+o(1)})$

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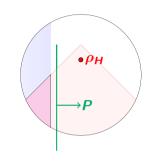
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Conclusion

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we established speedups for important problem class:

$$\label{eq:continuity} \begin{aligned} & \underset{\pmb{X} \in \mathbb{S}^n}{\text{maximize}} & & \operatorname{tr}(\pmb{A} \; \pmb{X}) \\ & \text{subject to} & & \operatorname{diag}(\pmb{X}) = \frac{1}{n} \pmb{1} \\ & & & \operatorname{tr}(\pmb{X}) = 1, \; \pmb{X} \succeq \pmb{0} \end{aligned}$$



our strategy:

- (i) replace optimization by a sequence of feasibility problems
- (ii) change of variables: $extbf{\textit{X}} \leftarrow
 ho_{ extbf{\textit{H}}} = rac{\exp(- extbf{\textit{H}})}{\operatorname{tr}(\exp(- extbf{\textit{H}}))}$
- (iii) iteratively penalize infeasible directions by Hamiltonian Updates $m{H} \leftarrow m{H} + \epsilon m{P}$
- (iv) boost runtime by preparing each ho_H on quantum computer

our result: we obtain approximate solutions faster than existing approaches:

$$\tilde{\mathcal{O}}(n^2s)$$
 (classical) and $\tilde{\tilde{\mathcal{O}}}(n^{1.5}\sqrt{s}^{1+o(1)})$ (quantum) vs. $\tilde{\tilde{\mathcal{O}}}(n^{2.5}s)$ (classical)

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Summary

- 1 improve runtime scaling in approximation accuracy ϵ
- 2 implementation on near-tearm devices or classical computers
- 3 adapt meta-algorithm to other important convex optimization problems:
 - quantum state tomography
 - semi-discrete matrix factorization

Thank you!