Quantum Approximate Optimization Algorithm

Landscape Independence

and

Sherrington Kirkpatrick Model

E.F. Google MIT

Combinatorial Optimization

n bits $\quad$ m clauses

$$C(z) = \sum_{\alpha=1}^{m} C_\alpha(z) \qquad z_1 \dots z_n = z$$

$$C_\alpha(z) = \begin{cases} 1 & \text{satisfies} \\ 0 & \text{does not} \end{cases}$$

$$C_{max} = \max_z C(z)$$

Want $\boxed{\dfrac{C(z)}{C_{max}}}$ big.

Quantum Algorithm Ingredients:

$$\underline{U(C,\gamma)} = e^{-i\gamma C} = \prod_{\alpha=1}^{m} e^{-i\gamma C_\alpha}$$

$$B = \sum_{j=1}^{n} X_j \qquad X_j = \sigma_j^x$$

$$\underline{U(B,\beta)} = e^{-i\beta B} = \prod_{j=1}^{n} e^{-i\beta X_j}$$

$$\underline{|s\rangle} = \frac{1}{\sqrt{2^n}} \sum_z |z\rangle$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

All easy to construct!

For any integer $p \geq 1$ $\underline{\gamma_1 \ldots \gamma_p = \vec{\gamma}}$ $\underline{\beta_1 \ldots \beta_p = \vec{\beta}}$

$$\underline{|\vec{\gamma}, \vec{\beta}\rangle} = U(B, \beta_p)\, U(C, \gamma_p) \cdots U(B, \beta_1)\, U(C, \gamma_1)\, |s\rangle$$

Required Circuit Depth at most $mp + p$.

$$F_p(\vec{\gamma}, \vec{\beta}) = \langle \vec{\gamma}, \vec{\beta} | C | \vec{\gamma}, \vec{\beta} \rangle$$

$$M_p = \max_{\vec{\gamma}, \vec{\beta}} F_p(\vec{\gamma}, \vec{\beta})$$

$$\boxed{M_p \geq M_{p-1}}$$

Can Show $\boxed{\lim_{p \to \infty} M_p = C_{max}}$

Quantum Algorithm with Angle Search

Fix p. Start with angles $(\vec{\gamma}, \vec{\beta})$

Use the Quantum Computer to make

$$|\vec{\gamma}, \vec{\beta}\rangle$$

Measure to get a string z and $C(z)$.

Repeat with same angles to get a good estimate of $F_p(\vec{\gamma}, \vec{\beta})$

Repeat with new angles to get near

$$M_p = \max_{\vec{\gamma}, \vec{\beta}} F_p(\vec{\gamma}, \vec{\beta})$$

For p fixed we can classically preprocess and determine the best angles in advance.

Example: Max Cut on 3-regular graphs

$$C = \sum_{\langle jk \rangle} C_{\langle jk \rangle}$$

$$C_{\langle jk \rangle} = \frac{1}{2}\left(-Z_j Z_k + 1\right) \qquad Z_j = \pm 1$$

p = 1 Look at contribution from edge $\langle jk \rangle$

$$\langle s | e^{i\gamma C} e^{i\beta B} Z_j Z_k e^{-i\beta B} e^{-i\gamma C} | s \rangle$$

$$|s\rangle = |+\rangle_1 |+\rangle_2 \cdots |+\rangle_n$$

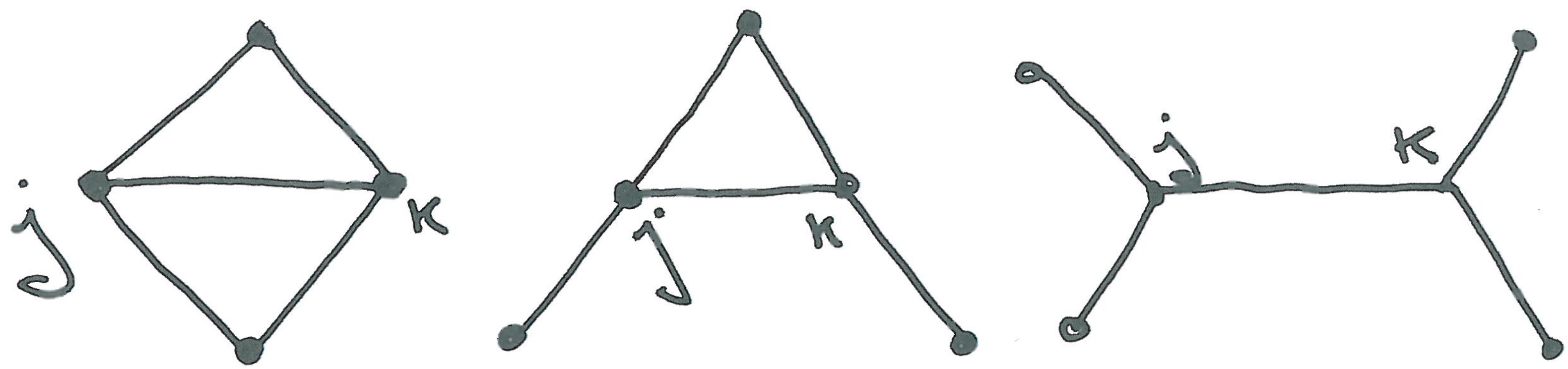
$$e^{i\beta B} Z_j Z_k e^{-i\beta B} = e^{i\beta(X_j + X_k)} Z_j Z_k e^{-i\beta(X_j+X_k)}$$

$$= [\cos 2\beta\, Z_j + \sin 2\beta\, Y_j][\cos 2\beta\, Z_k + \sin 2\beta\, Y_k]$$

only bits j k involved

Conjugate with $e^{i\gamma C}$

only bits connected to j, k involved



3 possible subgraphs

Each subgraph type gives a function of γ, β which does not depend on n or m.
These can be evaluated on a classical computer looking at a 4, 5 or 6 qubit system. Then

$$F_1(\gamma, \beta)$$

can be evaluated on a classical computer and optimal angles chosen.

At $p=1$, the QAOA will produce a cut that is at least $.6924$ times the optimal cut. For ALL instances of 3-regular Max Cut!

QAOA Typical Performance

Random Graphs with some distribution

e.g. * 3 regular graphs

or * each edge included with probability $3/(n-1)$

or * each edge included with probability $d/(n-1)$

or * whatever

Not Worst Case but Typical

We demonstrate that if we fix parameters $(\vec{\gamma}, \vec{\beta})$ then for typical instances

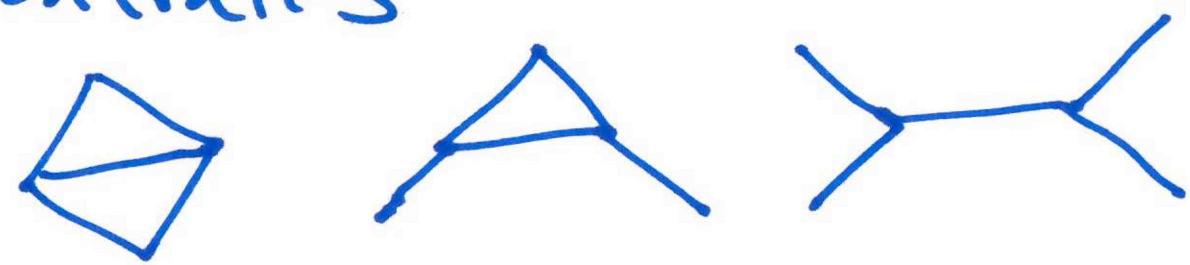
$$\left[F(\vec{\gamma}, \vec{\beta}) = \langle \vec{\gamma}, \vec{\beta} | C | \vec{\gamma}, \vec{\beta} \rangle \right]$$

is the same for all instances (up to $\sqrt{m}$ fluctuations)

F.G.S.L. Brandao, M. Broughton

S. Gutmann, H. Neven

Consider $p=1$, Max-Cut on 3 regular graphs, uniformly generated. For large n fraction of subgraph types concentrates



$\Rightarrow$ $\langle \gamma, \beta | C | \gamma, \beta \rangle$ is instance independent $n \to \infty$

20 vertex 3-regular graphs
25 samples

	Low		Random		High	
p	Mean	Std.	Mean	Std.	Mean	Std.
2	6.636	0.319	14.691	0.036	22.409	0.228
3	5.218	0.294	15.125	0.042	23.109	0.175
4	3.933	0.259	14.627	0.157	23.822	0.272
5	3.132	0.159	15.725	0.113	24.349	0.179
6	2.550	0.100	16.404	0.140	24.918	0.266
7	1.954	0.088	15.975	0.096	25.110	0.221

angles are not optimal

Why? $F(\vec{\gamma},\vec{\beta}) = \sum_{\alpha=1}^{m} \langle \vec{\gamma},\vec{\beta} | C_\alpha | \vec{\gamma},\vec{\beta} \rangle$

Fix $\vec{\gamma},\vec{\beta}$ $\quad F = \sum_{\alpha=1}^{m} C_\alpha \qquad 0 \leq C_\alpha \leq 1$

There is a distribution over instances.
Think of C_α as a random variable.

F is of order m.

If the terms are independent, then regardless of the distribution, by the Law of Large Numbers, the standard deviation is of order $\sqrt{m}$ $\Longrightarrow$ concentration!!!!

To test this we measured the correlation coefficients between different C_α on random ~~in~~stances. Always very small. Like 0.04

This was at 20 bits.

We bet that the correlation coefficients at higher bit number will also be small.

New Strategy for Random Instances

* Pick one instance.
* Work hard to get good angles.
  - maybe requires many calls to the Quantum Computer -
* Use these angles for other instances!

  Refine if needed.

Another Finding

Toss a random graph at 10 bits
p = 8 get good angles
Approximation ratio 0.984
Use these 16 angles at 24 bits
Get an average approximation ratio of 0.934

Leap Frog Strategy

Want to use Quantum Computer at 100 qubits to find good cut on a random 100 vertex graph.

Toss a 20 vertex graph.
Run classical computer at $p=6$ to get good angles

Toss a 50 vertex graph.
Run Quantum Computer with those angles. Refine

Go to 100 vertex graph with these angles.

Sherrington Kirkpatrick

n bits $z_a = \pm 1$

$$\left[C = \frac{1}{\sqrt{n}} \sum_{a<b} J_{ab} z_a z_b \right]$$

mean $J_{ab} = 0$ var $J_{ab} = 1$

As $n \to \infty$

$$\left[\frac{E_{max}}{n} = .763\ldots \right]$$

$$\langle s| e^{i\gamma C} e^{i\beta B} \frac{C}{n} e^{-i\beta B} e^{-i\gamma C} |s\rangle$$

$\boxed{\begin{matrix} p=1 \\ \text{QAOA} \end{matrix}}$

$$= \frac{1}{2^n} \sum_{z^1} \sum_{z^2} \sum_{z^3} e^{i\gamma C(z^1)} \langle z^1| e^{+i\beta B} |z^2\rangle \cdot$$

$$\frac{1}{n} C(z^2) \langle z^2| e^{-i\beta B} |z^3\rangle e^{-i\gamma C(z^3)}$$

$$C(z) = \frac{1}{\sqrt{n}} \sum_{a<b} J_{ab} z_a z_b$$

Now for ϕ small

$$\left[E_J e^{i\phi J} = (1 - \phi^2/2) + \cdots \right]$$

$$\left[E_J e^{i\phi J} J = i\phi + \cdots \right]$$

Upstairs $\gamma \sum_{a<b} J_{ab}(Z_a^1 Z_b^1 - Z_a^3 Z_b^3)$

Downstairs $\gamma \sum_{c<d} J_{cd} Z_c^2 Z_d^2$

Turn Crank

$$\left[E_J \langle \gamma,\beta | \frac{C}{n} | \gamma,\beta \rangle \text{ as } n \to \infty \right.$$

$$\left. = \underline{2 \sin 2\beta \cos 2\beta \; \gamma e^{-2\gamma^2}} \right]$$

Optimum at $\beta = \pi/8 \quad \gamma = 1/2$

get $\boxed{\frac{1}{2\sqrt{e}} \approx .303}$

For any fixed p we have an iterative procedure to evaluate

$$\boxed{\lim_{n\to\infty} E_J \langle \vec{\gamma}, \vec{\beta} | \tfrac{C}{n} | \vec{\gamma}, \vec{\beta} \rangle}$$

~~goes~~ takes $\sim (16)^p$ steps

optimized out to $p = 8$

$p=1$	.303
2	.407
⋮	⋮
8	.607

Again p fixed

$n \to \infty$

Also show

$$\lim_{n\to\infty} E_J\left[\langle\vec{\gamma},\vec{\beta}|\left(\frac{C}{n}\right)^2|\vec{\gamma},\vec{\beta}\rangle\right] = \lim_{n\to\infty} E_J^2\left[\langle\vec{\gamma},\vec{\beta}|\frac{C}{n}|\vec{\gamma},\vec{\beta}\rangle\right]$$

which proves Landscape Independence for the SK Model.

Can also show that for $n\to\infty$, for typical instances all measured strings z have

$$\left\{ \frac{C(z)}{n} \text{ close to } \langle\vec{\gamma},\vec{\beta}|\frac{C}{n}|\vec{\gamma},\vec{\beta}\rangle \right\}.$$

Want to show that for $n \to \infty$, and then $p \to \infty$ we achieve the Parisi value of .763...

Want to apply our techniques to other problems where instances are drawn from a distribution....

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