Successive minima-type inequalities

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Minkowski's successive minima

- Let $K \in \mathcal{K}^n = \{K \subset \mathbb{R}^n$ convex, compact and $\mathrm{int}\,(K) \neq \emptyset\}$, $K = -K$,

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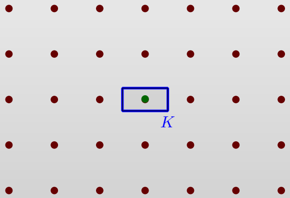

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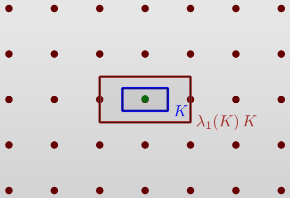

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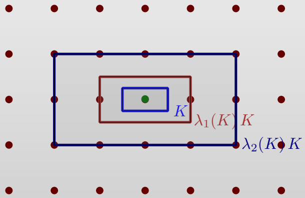

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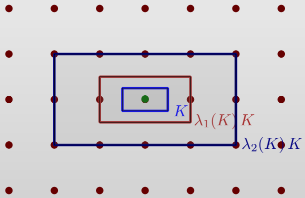

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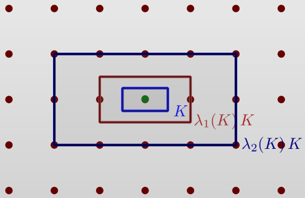
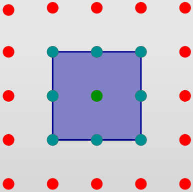


- $\lambda_1(K) \leqslant \lambda_2(K) \leqslant \ldots \leqslant \lambda_n(K)$.
- $\lambda_1(K) = \min\{|\boldsymbol{a}|_K : \boldsymbol{a} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}\}$.

- Minkowski's 1st theorem, 1896.

$$\operatorname{vol}(K) \leqslant 2^n \left(\frac{1}{\lambda_1(K)}\right)^n .$$



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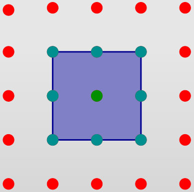
$$\mathrm{vol}\,(K) \leqslant 2^n \left(\frac{1}{\lambda_1(K)}\right)^n.$$

$$\Leftrightarrow \left[\mathrm{vol}\,(K) \geqslant 2^n \Rightarrow K \cap \mathbb{Z}^n \backslash \{0\} \neq \emptyset\right]$$



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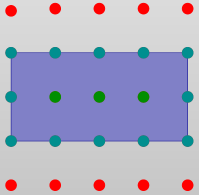

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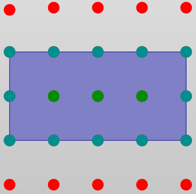

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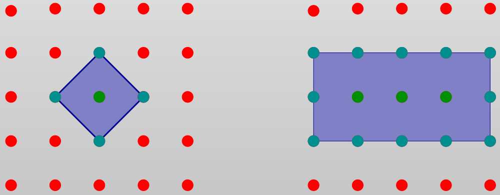

Extensions/Generalizations

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  - Via Brunn-Minkowski one gets for $K \in \mathcal{K}^n$

$$\frac{2^n}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(K_c)} \leqslant \operatorname{vol} K \leqslant 2^n \prod_{i=1}^{n} \frac{1}{\lambda_i(K_c)}.$$

- Let $K \in \mathcal{K}^n$ with centroid $\frac{1}{\mathrm{vol}\, K} \int_K \boldsymbol{x}\, \mathrm{d}\boldsymbol{x} = \mathbf{0}$.

$$\frac{n+1}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(K)} \leqslant \mathrm{vol}\, K \leqslant \frac{(n+1)^n}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(K)} \; ?$$

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  - Extremal cases would be $S^n_{\mathrm{min}} = \mathrm{conv}\,\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n,-\boldsymbol{1}\}$ and $S^n_{\mathrm{max}} = (n+1)\mathrm{conv}\,\{\boldsymbol{0},\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n\} - \boldsymbol{1}$.

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  - Upper bound implies the so-called Ehrhart-conjecture, 1964:

$$\mathrm{vol}\,(K) \geqslant \frac{(n+1)^n}{n!} \Rightarrow K \cap \mathbb{Z}^n \setminus \{\mathbf{0}\} \neq \emptyset.$$

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  By definition,

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  - $n = 3$, Woods, 1956.

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  Each of these inequalities implies the corresponding result for the volume.

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  - For $q_i = \lfloor \frac{2}{\lambda_i(K_c)} + 1 \rfloor$ let $p_i \geqslant q_i$ such that $p_{i+1} | p_i$. Then

$$\mathrm{G}(K) \leqslant \prod_{i=1}^{n} p_i.$$

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  - For ellipsoids.

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- Blaschke&Santalo, 1949; Bourgain&Milman, 1987; ...

$$\frac{c_1^n}{n!} \leqslant \mathrm{vol}\,(K)\mathrm{vol}\,(K^\star) \leqslant \frac{c_2^n}{n!}$$

  and it is conjectured that $c_1 = 4$; known to be true for $n = 2$ (Mahler, 1974) and for $n = 3$ (Iriyeh, Shibata, 2017+).

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- Banaszczyk, 1996.

$$1 \leqslant \lambda_{n-i+1}(K)\lambda_i(K^\star) \leqslant \mathrm{c}\, n(1 + \ln n).$$

- $$\mathrm{vol}\,(K) \leqslant 2^n \prod_{i=1}^{n} \lambda_i(K_c{}^\star) \text{ and } \mathrm{G}(K) \leqslant \prod_{i=1}^{n} (2\,\lambda_i(K_c{}^\star) + 1)\,.$$

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  - $K = -K$: $\boldsymbol{z}_i \in \lambda_i(K^\star)K^\star \cap \mathbb{Z}^n$, $1 \leqslant i \leqslant n$. Then

    $$K \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{z}_i, \boldsymbol{x} \rangle| \leqslant \lambda_i(K^\star),\, 1 \leqslant i \leqslant n\}.$$

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  - $K = -K$: $\boldsymbol{z}_i \in \lambda_i(K^{\star})K^{\star} \cap \mathbb{Z}^n$, $1 \leqslant i \leqslant n$. Then
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  - Mahler's "volume conjecture" implies this bound.
  - Even the weaker inequality
    $$\frac{2^n}{n!} \lambda_1(K^{\star})^n \leqslant \mathrm{vol}\,(K)$$
    is not known for $n \geqslant 4$.

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- H., Xue, 2019. For $n = 2$

$$\frac{3}{2}\lambda_1(K_c{}^\star)\lambda_2(K_c{}^\star) + \frac{1}{2}\lambda_1(K_c{}^\star)\Big(\lambda_2(K_c{}^\star) - \lambda_1(K_c{}^\star)\Big) \leqslant \mathrm{vol}\,(K).$$

Packing minima

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Packing minima

- based on ongoing joint work with Matthias Schymura (EPFL) and Fei Xue (TU Berlin).
- For a lattice $\Lambda \subset \mathbb{R}^n$ let

$$\lambda_i(K, \Lambda) = \min\{\lambda > 0 : \dim(\lambda K \cap \Lambda) \geqslant i\}.$$

- For $0 \leqslant k \leqslant n$ let $\mathcal{L}_{\mathbb{Z}}(k)$ be the set of all k-dimensional linear lattice subspaces of $\mathbb{Z}^n$, i.e., $L \in \mathcal{L}_{\mathbb{Z}}(k)$ if $L = \text{lin}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_k\}$ with $\boldsymbol{a}_i \in \mathbb{Z}^n$ linearly independent.

- For $1 \leqslant i \leqslant n$

$$\rho_i(K) = \max\left\{\lambda_1\left(K_c|L^{\perp}, \mathbb{Z}^n|L^{\perp}\right) : L \in \mathcal{L}_{\mathbb{Z}}(n-i)\right\}.$$

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- Hence,

$$\rho_i(K) \leqslant \lambda_{n-i+1}(K_c),$$

  and $\rho_n(K) = \lambda_1(K_c)$.

- Since for $L \in \mathcal{L}_{\mathbb{Z}}(n-i)$

$$\lambda_i([K_c|L^{\perp}]^{\star}, [\mathbb{Z}^n|L^{\perp}]^{\star})\lambda_1\left(K_c|L^{\perp}, \mathbb{Z}^n|L^{\perp}\right) \geqslant 1$$

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we get

$$\lambda_1\left(K_c|L^{\perp}, \mathbb{Z}^n|L^{\perp}\right) \geqslant \frac{1}{\lambda_i(K_c{}^{\star} \cap L^{\perp}, \mathbb{Z}^n \cap L^{\perp})}.$$

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- Hence,

$$\rho_i(K) \geqslant \frac{1}{\min\{\lambda_i(K_c{}^\star \cap \tilde{L}, \mathbb{Z}^n \cap \tilde{L}) : \tilde{L} \in \mathcal{L}_{\mathbb{Z}}(i)\}} = \frac{1}{\lambda_i(K_c{}^\star)}$$

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and $\rho_1(K) = 1/\lambda_1(K_c{}^\star)$.

- For $1 \leqslant i \leqslant n$

$$\lambda_{n-i+1}(K_c) \geqslant \rho_i(K) \geqslant \frac{1}{\lambda_i({K_c}^\star)},$$

where the upper bound is always attained for $i = n$ and the lower bound for $i = 1$.

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- $$\frac{\mathrm{c}i(1+\ln(i))}{\lambda_i({K_c}^\star)} \geqslant \rho_i(K) \geqslant \frac{\lambda_{n-i+1}(K_c)}{\mathrm{c}(n+1-i)(1+\ln(n+1-i))}$$

- $$\frac{2^n}{n!}\prod_{i=1}^{n}\frac{1}{\rho_i(K)} \leqslant \mathrm{vol}\,(K) \leqslant 2^n\prod_{i=1}^{n}\frac{1}{\rho_i(K)}.$$

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- $$\mathrm{G}(K) \leqslant \prod_{i=1}^{n}\left\lfloor\frac{2}{\rho_i(K)}+1\right\rfloor.$$
- The bound is stronger than the one with $\lambda_i({K_c}^\star)$, but, of course, weaker than the conjectured one with $\lambda_i(K_c)$.

Thank you for your attention!