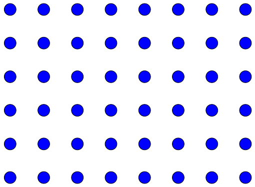
AGSPs and an Area Law for Gapped 1D Systems

Itai Arad, Alexei Kitaev, Zeph Landau, Umesh Vazirani

The difficulty of understanding many-body physics



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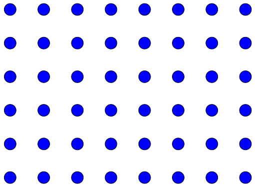

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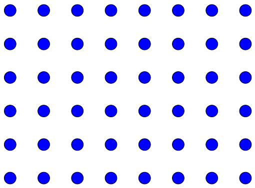

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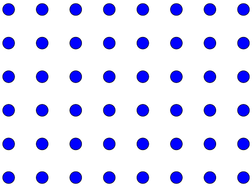

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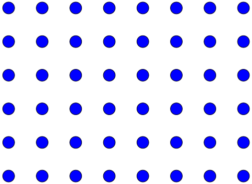

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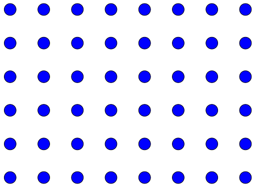

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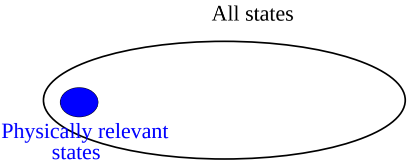
Exponential Dimensional Space

So even describing a state requires exponential amount of information.

A Basic Question



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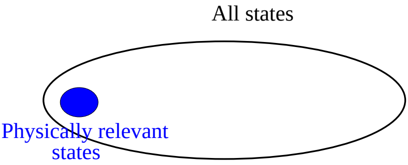
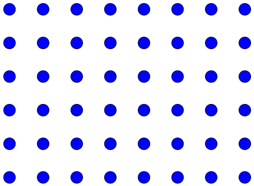


Can we develop a better understanding of a class of relevant states?

- Do they have a special structure?
- Does that structure allow for meaningful short descriptions?
- Does that structure allow us to compute various properties of them?

Physically Relevant States: Ground States of Local Hamiltonians



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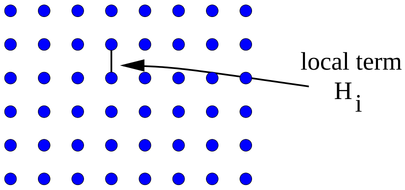

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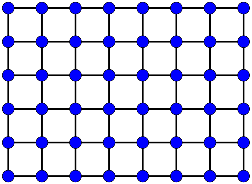

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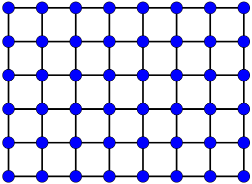

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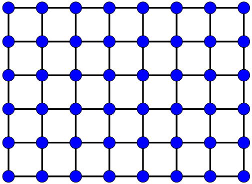

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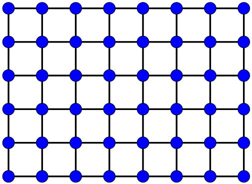

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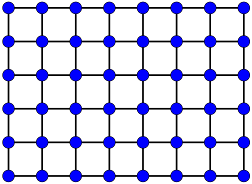
- Gap = distance between the lowest two eigenvalues.
- Focus on unique ground state and constant gap.

Ground states model the state of the system at low temperatures.

The Fundamental Quest: understanding ground states



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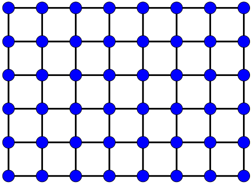

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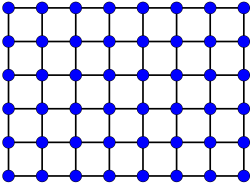

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Area Law formulation

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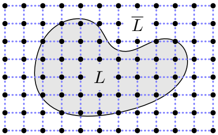

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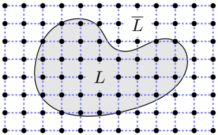

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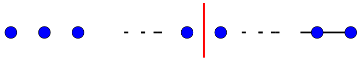
['01, Vidal, Latorre, Rico, Kitaev] Area Law formalized in terms of entanglement entropy.

Area Law in 1D systems



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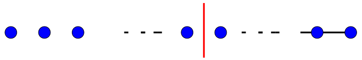

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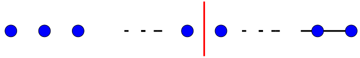

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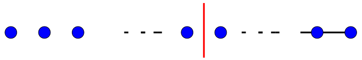

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- Does it suggest an algorithm for finding the ground state?

The birth of Approximate Ground State Projections

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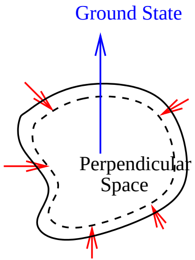
How to generalize this idea?

Approximate Ground State Projection (AGSP)



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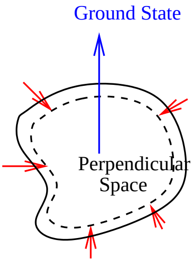

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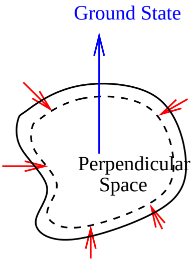

Properties:

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- It isn't too complex.

Consequences of AGSPs

Two new results:

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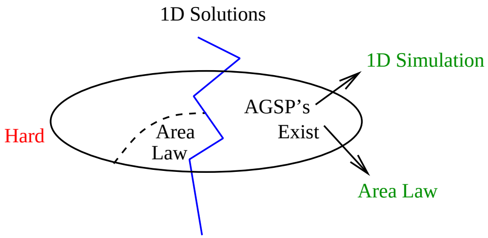
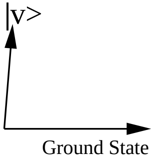


An area law in 2 steps

Area law proof:

1. Find a not very complex state that has constant overlap with the ground state.



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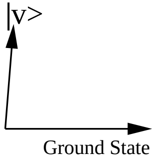

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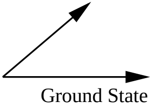

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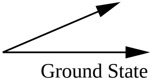

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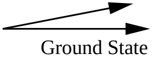

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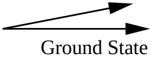

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Both steps use AGSPs– the first is much more delicate.

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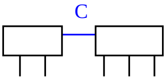

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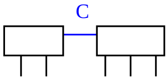

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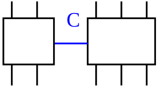

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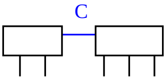

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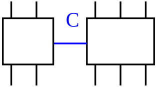

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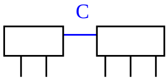

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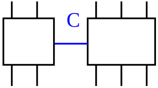

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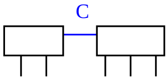

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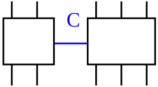

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- Additive for sums of states or operators.

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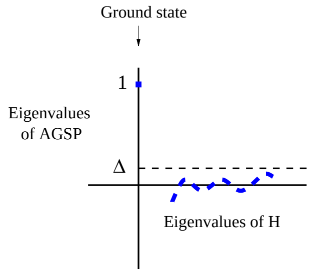

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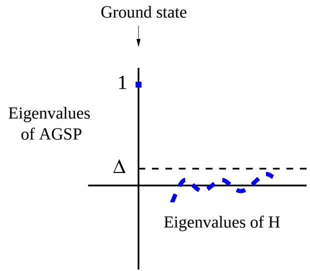

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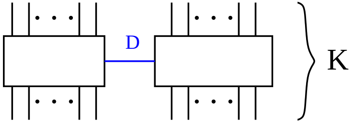

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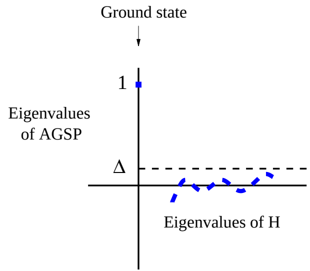

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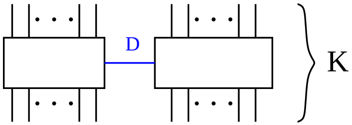

Critical threshold $D\Delta < 1$.

Role of AGSP in proving Area Law cont.

Theorem (Area Law) [Arad, Landau, Vazirani] The existence of an AGSP K for which $D\Delta < 1/2$ proves that the ground state has entropy $O(\log D)$.

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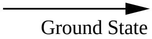

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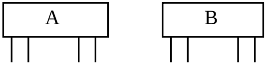

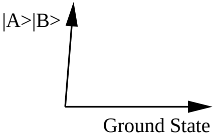

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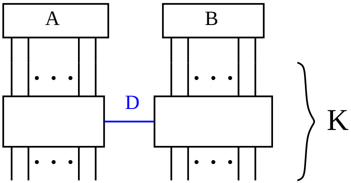

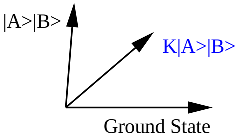

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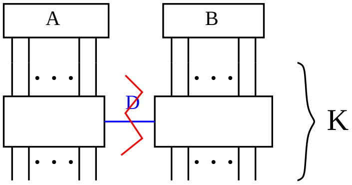

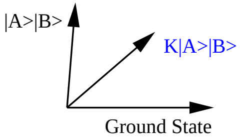

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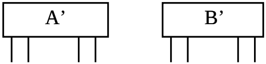

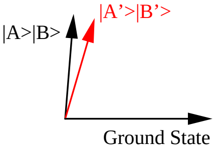

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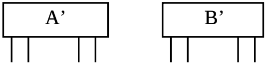

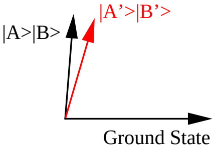

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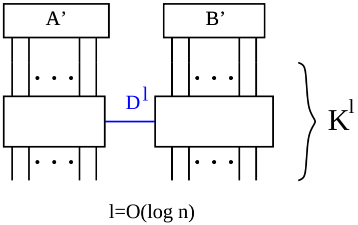

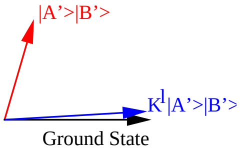

AGSP construction

AGSP will be a well chosen polynomial in the local terms H_i.

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- Analysis of the entanglement rank will involve **polynomial interpolation**.

Building Intuition

H has eigenvalues in $[0, n]$. So $H/||H||$ has eigenvalues in $[0, 1]$.

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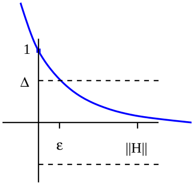

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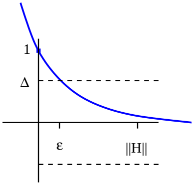

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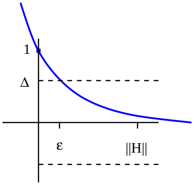

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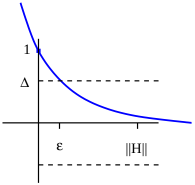

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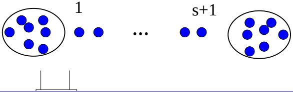
- Smaller $||H||$ would be better but we don't want to lose the 1D structure of H $\rightarrow$ truncate the ends to get $H' = (H_L + H_1 + H_2 + \cdots + H_s + H_R)$.



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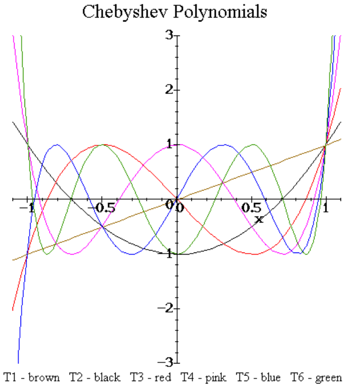
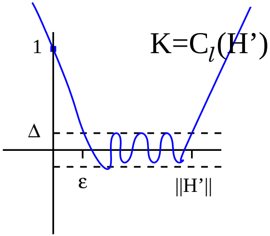
Chebyshev polynomials: small in an interval:



Building intuition: using Chebyshev polynomials

Candidate 2: $C_\ell(H')$= dilation and translation of Chebyshev applied to H':



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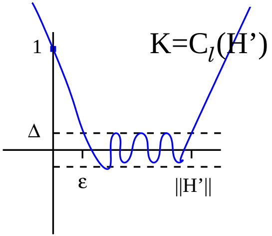

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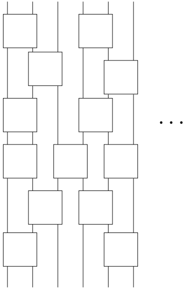
This will be our AGSP. How complex is it?

AGSP complexity: Entanglement rank analysis

$$(H')^\ell = \sum(\text{ product of } H_j).$$

For a single term:



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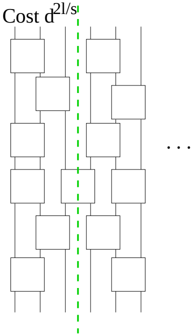

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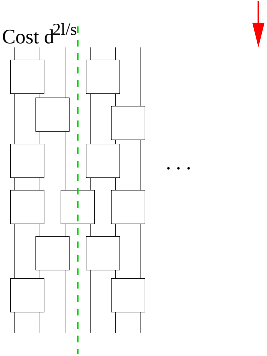

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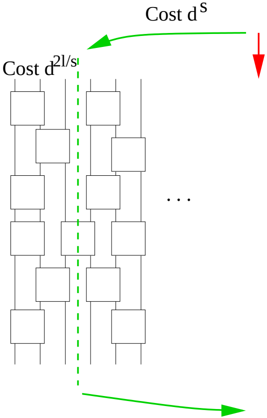

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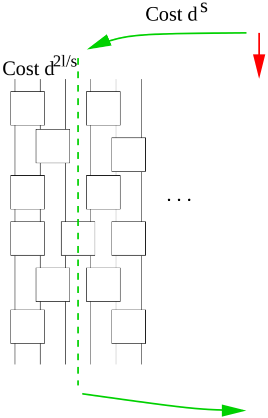
Total: $d^{2\ell/s+s}$



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Problem: Too many (s^ℓ) terms in naive expansion of $(H')^\ell$.

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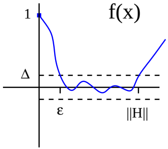
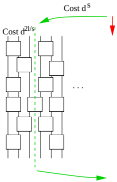
Need to group terms in a nice way (polynomial interpolation) but it all works out with total entanglement increase of the same order as the single term.

Putting things together: Area Law for H'

Chebyshev $C_\ell(H')$ has $\Delta \approx e^{-O(\ell/\sqrt{s})}$:



Entanglement analysis yields $D \approx O(d^{\ell/s+s})$.



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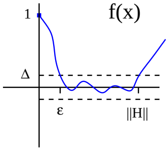

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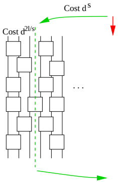

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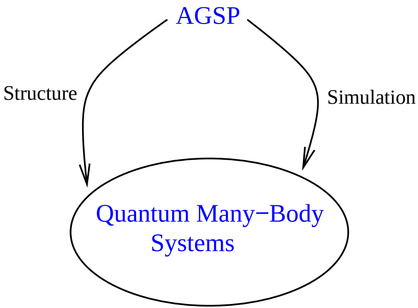
Area Law of entanglement entropy $\log(D) = \tilde{O}(\frac{\log^3(d)}{\epsilon})$

The Landscape



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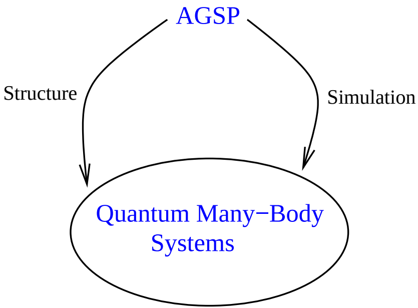

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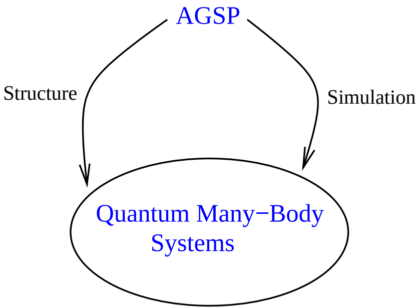

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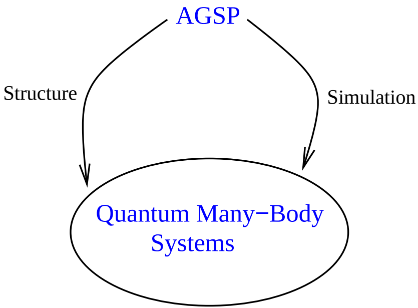

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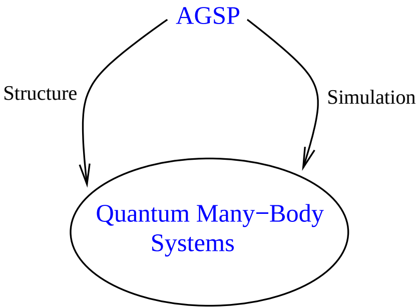

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- Of independent interest: robustness theorem of truncation.