**ECC and HDX III**

# Random Walks on High Dimensional Expanders

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In this talk we will redefine high dimensional expanders through random walks. This definition will give us a new perspective on high dimensional expansion, and with it we will explore other properties that weren't apparent when we looked at spectral expansion in the links. More specifically, this new definition will allow us to do Boolean function analysis, for functions on $k$-faces: $f : X(k) \to \{\pm 1\}$.

## 1 Natural Random Walks

Let $X$ be a $d$-dimensional simplicial complex. We define two natural random walks on the $k$-faces.

1. The upper walk where given $t \in X(k)$ we choose $t'$ by a shared $k+1$-face:
   - Sampling $s \in X(k+1)$ given that $s \supset t$.
   - Choosing $t' \subset s$.
2. The lower walk where given $t \in X(k)$ we choose $t'$ by a shared $k-1$-face:
   - Sampling $r \in X(k-1)$ given that $r \subset t$.
   - Choosing $t' \supset r$.

**Example 1.1.** *Consider the simplicial complex in 1. We can draw the* simplicial complex diagram*, as in 2.*

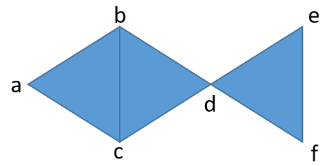


Figure 1: Just your day-to-day simplicial complex.

we denote the upper walk's adjacency operator by $D_{k+1}U_k$ and the lower walk's adjacency operator by $U_{k-1}D_k$. Why this notation?

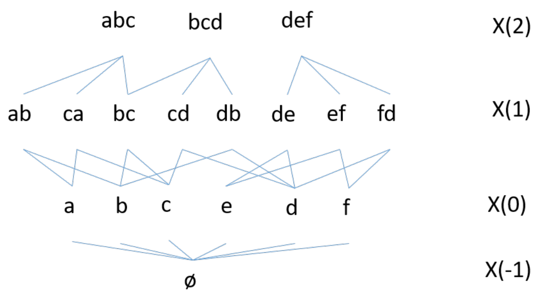


Figure 2: The simplicial complex diagram.

## 2 The Up and Down Operators

Simple. As we can see in the diagram, and in the description of the two walks. These walks consist of two steps - the up step and the down step.

These steps corespond to the following two averaging transformations

**Definition 2.1** (Up and Down Operators)**.** *1. The* Up *operator, that takes* $f : X(k) \to \mathbb{R}$ *and returns* $U_k f : X(k+1) \to \mathbb{R}$*, by*

$$U_k f(s) = \mathop{\mathbb{E}}_{t \subset s}[f(t)] = \frac{1}{k+2} \sum_{t \subset s} f(t).$$

*2. The Down operator, that takes* $f : X(k) \to \mathbb{R}$ *and returns* $D_k f : X(k-1) \to \mathbb{R}$ *by*

$$D_k f(r) = \mathop{\mathbb{E}}_{t \supset r}[f(t)],$$

*where the expectation is by choosing* $s_k = t$ *given that* $s_{k-1} = r$*. Here we don't necessarily have a clean expression as in the up operator.*

**Remark 2.2.** *1. Note that the down operator is adjoint to the up operator* $U_k^* = D_{k+1}$*.*

*2. By different compositions of these operators we can define different random walks, such as* $DDUU$ *corresponds to the walk where we go from k-face to k-face by a shared* $(k+2)$*-face.*

**Example 2.3.** *If* $\mathbb{1}_B : X(1) \to \{0,1\}$ *is an indicator of some edges. then*

$$U_1 \mathbb{1}_B(s) = \frac{|\{e \subset s : e \in B\}|}{3},$$

$$D_1 \mathbb{1}_B(v) = \frac{Pr\left[B \cap X_v(0)\right]}{Pr\left[v\right]}.$$

We abbreviate by

$$U_{j \nearrow i} = U_{i-1} \dots U_j,\ D_{i \searrow j} = D_{j+1} \dots D_{i-1} D_i.$$

## 3 High Dimensional Expanders through Random Walks

In the example, these two walks are not the same. when can we say that they are similar?

**Example 3.1.** *In a graph (which we think of as* $G = (V = X(0), E = X(1))$*), the upper walk is the* $\frac{1}{2}$*-lazy version of the usual random walk from vertex to vertex, namely* $DU_0 = \frac{1}{2} I + \frac{1}{2} A_G$*.*

*The lower walk, is the walk where given a vertex* $v$ *we choose the other vertex* $u$ *independently. Thus* $U_{-1} D_0 f(v) \cong \mathop{\mathbb{E}}_{u \in V}[f(u)]$ *the constant function.*

*In a graph, when is the adjacency walk "similar" to the independent walk? Precisely when* $G$ *is an expander graph (at least in a spectral sense)! Infact,*

$$\lambda(G) = \left\| A_G - U_{-1} D_0 \right\|_{op}.$$ [1]

*This is because* $(A_G - UD)\mathbb{1} = 0$ *and for every* $f \perp \mathbb{1}$, $(A_G - UD) f = A_G f$*.*

### 3.1 Laziness

In general, the laziness probability (i.e. that $s \in X(k)$ transitions to $s$), doesn't depend on the specific structure of $X$, it is always $\frac{1}{k+2}$ (why?).

[1] $\|B\|_{op} = \sup_{\|f\|=\|g\|=1} \langle Bf, g \rangle$. In this case, it can be shown that it's the maximal eigenvalue in absolute value.

Thus $DU$ is a convex combination of the identity, and a non-lazy upper walk operator

$$D_{k+1}U_k = \frac{1}{k+2}I + \frac{k+1}{k+2}M_k.$$

The lower walk doesn't have a constant laziness, and in most interesting examples, the probability of laziness is negligible.

# 4 High Dimensional RW-Expanders

We are ready to define a Random-Walk High Dimensional Expander:

**Definition 4.1** (RW-HDX)**.** *Let $X$ be a pure $d$-dimensional simplicial complex. We say $X$ is a $\lambda$-Random Walk High Dimensional Expander if for all $0 \leq k \leq d-1$:*

$$\|M_k - UD_k\|_{op} \leq \lambda.$$

Morally, this means that the non-lazy upper walk, and the lower-walk are similar (in a spectral sense). Two questions naturally arise:

1. Do these objects exist? (at least non-trivially)
2. Why is this interesting?

The answer to the first question is yes. We have an equivalence theorem between two-sided local link expanders, and RW-HDX:

**Theorem 4.2** ([Dik+18] Theorem 5.2)**.** *Let $X$ be a pure $d$-dimensional simplicial complex.*

1. *If $X$ is a $\lambda$-two-sided link expander. Then $X$ is a $\lambda$-RW-HDX.*
2. *If $X$ is a $\lambda$-RW-HDX. Then $X$ is a $3(d+1)\lambda$-two-sided link expander.*

# 5 Proof of the Equivalence Theorem

*Proof of item 1.* Let's take as a representative case $k=1$, the general proof is the same. The proof is via Garland's method. We can calculate $\langle M_k f, g\rangle$ by local forms:

$$\langle M_k f, g\rangle = \mathop{\mathbb{E}}_{t\sim t'}[f(t)g(t')] = \mathop{\mathbb{E}}_{r\in X(k-1)}\mathop{\mathbb{E}}_{vu\in X_r(1)}[f(rv)g(ru)].$$

The last equality is due to the fact that choosing an edge in the upper walk can beconditioned on first choosing the intersection $r = t\cap t'$, and given that intersection, to choose (directed) edge in it's link. We define the localization $f_r, g_r : X_r(0) \to \mathbb{R}$ by $f_r(v) = f(rv)$. Then

$$= \mathop{\mathbb{E}}_{r}[\langle A_r f_r, g_r\rangle_{X_r}].$$

Similar to yesterday, we decompose $f_r = f_r^0 + f_r^\perp$ to the constant part and the part that is perpendicular to it (and the same for $g$.

$$\mathop{\mathbb{E}}_{r}[\langle A_r f_r, g_r\rangle_{X_r}] = \mathop{\mathbb{E}}_{r}[\langle f_r^0, g_r^0\rangle] + \mathop{\mathbb{E}}_{r}[\langle A_r f_r^\perp, g_r^\perp\rangle].$$

The right part of the sum is bounded by $\pm\lambda$. The constant $f_r^0 = Df(r)$. Thus

$$\mathop{\mathbb{E}}_{r}[\langle f_r^0, g_r^0\rangle] = \langle D_k f, D_k g\rangle = \langle UDf, g\rangle,$$

or

$$\|M_k - UD\| \le \lambda.$$

□

The second item might be more surprising, How does the fact that the global operators $M_k$ and $UD$ are similar in operator norms indicate that in every link (no matter how small), the link is an expander?

The operator norm is an expression with a maximum over all vectors, we can use this bound on vectors that are concentrated on specific links, to show that the adjacency operators of the links also have small norm.

# 6 The Upper Walk - an (approximate) spectral decomposition

In the remainder of this talk, we'll try to decompose functions $f : X(k) \to \mathbb{R}$ in a way that generalizes the trivial (yet useful) decomposition of $f = \alpha\mathbb{1} + f^{\perp}$. We can understand (up to an error) what the graph operator does to $f$. It leaves $\alpha\mathbb{1}$ as is, and "kills" $f^{\perp}$.

The notion of decomposition of functions to simpler components, should remind us the Fourier decomposition on the Boolean Hypercube.

One natural way to define these simpler components, is by the the levels of the simplicial complex.

**Example 6.1.** *Fix some $e_0 \in X(1)$. A function*

$$f(s) = \mathbb{1}_{\{s:e_0 \subset s\}}$$

*should be considered as a function that "comes from" the edges $X(1)$, because:*

1. *The output of the function depends only on whether $s$ contains an edge or not.*
2. *Since*
$$f = cU_{1\nearrow k}\mathbb{1}_{e_0},$$
*for some constant $c$.*

The first reason is a combinatorial property, that is hard to encapsulate. In particular, it is not clear why this defines a subspace of functions.

The second idea has more promise. We can define functions of level $j$, as $U_{j\nearrow k}h_j$. One obvious problem arises though - how can we be sure that $h_j$ is not itself $h_j = Ug_{j-1}$?

This problem is solvable. We ask for $h_j$ to be orthogonal to all possible $Ug_{j-1}$ (which is equivalent to $Dh_j = 0$). Indeed, it can be proven that we can write any $f : X(k) \to \mathbb{R}$ as

$$f(s) = \sum_{j=-1}^{k} U_{j\nearrow k}h_j$$

for $h_j$ as above. Namely, each $U_{j\nearrow k}h_j$ is the component of the function that comes from the $j$-th level.

However, this is a weak decomposition. It is not necessarily orthogonal. In fact, it is not necessarily even unique. However, when $X$ is a $\lambda$-RW-HDX for a small enough $\lambda$ we *do* have these properties. Formally here's our theorem:

**Theorem 6.2** ([Dik+18])**.** *Let $f : X(k) \to \mathbb{R}$ be any function. Then we can write*

$$f(s) = \sum_{j=-1}^{k} f^{=j}(s).$$

*Where $f^{=j} = U^{j\nearrow k}h_j$ for some $h_j \in \ker D_j$. When $X$ is a $\lambda$-high dimensional expander, where $\lambda < \frac{1}{d+1}$, the following properties hold:*

1. *This decomposition is unique, (we call this property a* proper *simplicial complex).*

2. *These vectors are the $UD$-operator's "approximate" eigenvectors.*

$$\left\| DUf^{=j} - \frac{k-j+1}{k+1} f^{=j} \right\| \le O(\lambda) \left\| f^{=j} \right\|.$$

3. *For $j \neq i$, these functions are almost orthogonal:*

$$\left\langle f^{=j}, f^{=j'} \right\rangle \le O(\lambda) \left\| f^{=j} \right\| \left\| f^{=j'} \right\|.$$

# References

[Dik+18] Yotam Dikstein et al. "Boolean function analysis on high-dimensional expanders". In: *Proc. 20th International Workshop on Randomization and Computation (RANDOM)*. Ed. by Eric Blais et al. Vol. 116. LIPIcs. Schloss Dagstuhl, 2018. DOI: 10.4230/LIPIcs.APPROX/RANDOM.2018.37. eprint: 1804.08155.