ECC and HDX II

Oppenheim's Trickling Down Theorem

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This talk continues Irit Dinur's talk about High Dimensional Expanders. In the following hour we will prove the "Trickling Down Theorem" by [Opp18].This theorem demonstrates the power of "Garland's Method", where we calculate a global invariant of the simplicial complex with its links. This method allows us to deduce global behaivior out of local behaivior.

1 Spectral Expansion Trickles Down

Theorem 1.1 (Oppenheim)**.** *Let X be a d-dimensional simplicial complex for $d \geq 2$. Suppose that its one-dimensional skeleton (i.e. the graph $(X(0), X(1))$) is connected. Furthermore, assume that $\forall v \in X(0)$, X_v is a λ-one-sided spectral expander. Then the underlying graph of X is a μ-one sided spectral expander for $\mu = \frac{\lambda}{1-\lambda}$.*

Why the "Trickling Down Theorem"?

Suppose X is a $d = 3$-dimensional simplicial complex. Assume that the links of the edges $(X(1))$ are good spectral expanders. Then we can apply the theorem on the links of the vertices, to get that every link of a vertex is a good spectral expander. Then we apply it again to get that the underlying graph of the whole simplicial complex is a good spectral expander. The expansion "trickles down" from the edges to the whole complex.

We formalize this in the following corollary:

Corollary 1.2. *Let X be a d-dimensional simplicial complex such that the 1-skeleton of every link (including the entire simplicial complex) is connected. Assume that $\forall v \in X(d-2)$ X_v is a one-sided λ-expander. Then X is a μ-expander where $\mu = \frac{\lambda}{1-(d-1)\lambda}$.*

Remark 1.3. *For negative eigenvalues we have an analogous bound:*

Theorem 1.4 (Trickling Down for Negative Eigenvalues)**.** *Let X be as above. Assume that for all $v \in X(0)$, the smallest eigenvalue of A_v is $\geq \eta$. Then the smallest eigenvalue of the one dimensional skeleton is lower bounded bounded by $\frac{\eta}{1-\eta}$.* [1]

This might seem less interesting, but this has meaning even when $\eta = -1$, since when $\eta \leq 0$ then $\frac{\eta}{1-\eta} \geq \eta$. I.e. the negative eigenvalues become smaller when you look at lower level faces.

2 Applications

A priori to show that X is a high dimensional expander, one needs to check all the links. This theorem shows that good spectral expansion in the top level links of the simplicial complex, already implies good (under weak assumptions of connectivity). This is a local to global phenomenon.

Oppenheim's trickling down theorem power is that by analyzing the expansion of the links of co-dimension 2, and showing that the remaining links are connected (which is usually much easier), we get a bound on the link expansion of all the links that goes to 0 as the original expansion goes to 0 (as long as the dimension is fixed).

1. The analysis of Kaufman-Oppenheim Algebraic construction of High Dimensional Expanders.

[1]For this part, we do not require the underlying graph being connected.

2. An immediate construction of two-sided spectral expanders from one-sided spectral expanders: just take a d-dimensional one-sided link expander, and take a lower-level skeleton (i.e. take $Y = \bigcup_{j=-1}^{k} X(j)$ for $k < d$). This was already discovered by Dinur-Kaufmann, but this gives a one-line proof.

3 Proof of the Theorem

why is this theorem true?

When we have a *disconnected* underlying graph, then no-matter how well connected are the links, they cannot "feel" the disconnectedness of the graph. However, if a graph has a sparse cut, then at least one link will see that the graph has a sparse cut.

Surprisingly, though, the actual proof doesn't go through this line of reasoning. Instead, we use an algebraic point of view. We study the spectrum of the adjacency operator using the spectra of the links.

3.1 Some Preliminaries

The proof of this theorem only uses some basic facts from linear algebra and probability.

First we recall the adjacency operator of a graph:

Definition 3.1 (adjacency operator)**.** *The adjacency operator of the graph is an operator who takes as input $f : V \to \mathbb{R}$ and returns $Af : V \to \mathbb{R}$ given by*

$$Af(v) = \mathop{\mathbb{E}}_{u \sim v}[f(u)].$$

The probability of choosing u is taken by the probability to choose the edge $\{v, u\}$ given that we chose an edge containing v.

The trivial eigenvector is the all ones vector since $A\mathbb{1} = \mathbb{1}$. We say a graph is a λ-*one sided spectral expander* if all non-trivial eigenvalues of A are $\leq \lambda$. Note that this is a different (but equivalent) definition than the definition by the laplacian.

Recall the inner product that we defined previously. For real-valued functions on the vertices $f : V \to \mathbb{R}$.

$$\langle f, g \rangle = \mathop{\mathbb{E}}_{v \in V}[f(v)g(v)].$$

The adjacency operator has a special relation with this inner product:

$$\langle f, Ag \rangle = \mathop{\mathbb{E}}_{uv \in E}[f(u)g(v)].$$

Here are some more basic facts on the adjacency operator, all of which are elementary:

1. the constant all ones vector $\mathbb{1}$ is always an eigenvector of A and it's eigenvalue is 1.

2. The graph is connected if and only if all other eigenvalues are strictly less than 1.

3. A is self-adjoint, so if we decompose any function $f = \alpha\mathbb{1} + f^{\perp}$ where the two parts are perpendicular, then

$$\langle Af, f \rangle = \langle \alpha\mathbb{1}, \mathbb{1} \rangle + \left\langle Af^{\perp}, f^{\perp} \right\rangle;\ \alpha = \mathop{\mathbb{E}}_{u \in V}[f(u)].$$

4. When G is a λ-one sided spectral expander, then we can say that for any $f \perp \mathbb{1}$

$$\langle f, Af \rangle \leq \lambda \langle f, f \rangle.$$

3.2 The Proof

We prove that part about the positive eigenvalues, the negative eigenvalue part is similar.

Proof of Theorem 1.1. First we give some notation: Let A be the adjacency operator associated with the 1-skeleton $(X(0), X(1))$. Let A_v be the adjacency operator associated with X_v.

We begin with the following important claim

Claim 3.2 (Localization)**.** *Let $f, gX(0) \to \mathbb{R}$ be any real valued functions, and for $v \in X(0)$ denote by f_v, g_v their restrictions to X_v. Then*

$$\langle f, g \rangle = \mathop{\mathbb{E}}_{v \in X(0)} [\langle f_v, g_v \rangle_{X_v}], \tag{3.1}$$

and

$$\langle Af, g \rangle = \mathop{\mathbb{E}}_{v \in X(0)} [\langle A_v f_v, g_v \rangle_{X_v}], \tag{3.2}$$

This is what is called "Garland's Method": calculating global forms via local links.

Proof. For (3.1):

$$\langle f, g \rangle = \mathop{\mathbb{E}}_{u \in X(0)} [f(u) g(u)].$$

How do we choose a vertex $u \in X(0)$? We choose an edge $uv \in X(1)$ and take one of its vertices. We use the law of conditional expectation on the *other, non-chosen* vertex. In other words, we can choose this u by first choosing $v \in X(0)$, then choosing uv conditioned on v, and then take u. This is the same probability of choosing v and then u in the link of v.

For (3.2) we do the same:

$$\langle f, g \rangle = \mathop{\mathbb{E}}_{uw \in X(1)} [f(u) g(w)],$$

and we choose uw by choosing $v \in X(0)$ and then the triangle vuw (which is the same as choosing uw in the link of v. □

Suppose $f : X(0) \to \mathbb{R}$ is an eigenfunction with eigenvalue γ, and assume $f \perp \mathbb{1}$. Also assume $\langle f, f \rangle = 1$. We have by 3.2:

$$\gamma = \langle Af, f \rangle = \mathop{\mathbb{E}}_{v \in X(0)} [\langle A_v f_v, f_v \rangle], \tag{3.3}$$

Where f_v be the restriction of f to $X_v(0)$.

By assumption X_v is a one-sided λ-spectral expander and so the second largest eigenvalue of A_v satisfies $\leq \lambda$. Thus for any function $g : X_v(0) \to \mathbb{R}$ satisfying $g \perp \mathbb{1}$ we have by the spectral decomposition of A that

$$\langle Ag, g \rangle \leq \lambda \left\| g \right\|^2.$$

We assumed that $f \perp \mathbb{1}$, so if it were true that $f_v \perp \mathbb{1}_v$, then

$$\langle f_v, A_v f_v \rangle \leq \lambda \langle f_v, f_v \rangle,$$

and then we would get that $\gamma \leq \lambda$.

Unfortunately, the fact that $f \perp \mathbb{1}$ globally, doesn't necessarily mean that $f_v \perp \mathbb{1}_v$. It may be that on some vertices f_v has a large and positive weight, and on some, it has large negative weights.

Thus we define $f_v = \alpha_v \mathbb{1}_v + f_v^{\perp}$ a decomposition to the constant part, and the part perpendicular to it. And

$$\langle f_v, A_v f_v \rangle = \langle \alpha_v \mathbb{1}_v, \alpha_v \mathbb{1}_v \rangle + \left\langle f_v^{\perp}, A_v f_v^{\perp} \right\rangle.$$

As for the perpendicular part, we know that

$$\left\langle f_v^{\perp}, A_v f_v^{\perp} \right\rangle \leq \lambda \left\langle f_v^{\perp}, f_v^{\perp} \right\rangle$$

Hence we can write

$$(1-\lambda)\left\langle \alpha \mathbb{1}, \alpha \mathbb{1} \right\rangle + \lambda \left\langle f_v, f_v \right\rangle .$$

The expectation of the right part just amounts to $\lambda \langle f, f \rangle = \lambda$. In total, after taking expectation we get:

$$\gamma \leq (1-\lambda) \mathop{\mathbb{E}}_{v \in X(0)} [\alpha_v^2] + \lambda.$$

What is α_v? Recall that α is just the projection of f to $\mathbb{1}$, i.e.

$$\alpha = \langle f_v, \mathbb{1}_v \rangle = \mathop{\mathbb{E}}_{u \in X_v(0)} [f(u)].$$

This should look familiar - $u \in X_v(0)$ iff $u \sim v$. This means that $\alpha_v = Af(v)$. And we get that

$$\mathop{\mathbb{E}}_{v \in X(0)} [\alpha_v^2] = \mathop{\mathbb{E}}_{v \in X(0)} [Af(v)^2] = \langle Af, Af \rangle = \gamma^2.$$

We get that

$$\gamma \leq (1-\lambda) \mathop{\mathbb{E}}_{v \in X(0)} [\alpha_v^2] + \lambda.$$

With some more manipulations

$$\gamma(1-\gamma) \leq \lambda(1-\gamma^2).$$

As $\gamma < 1$ since the underlying graph is connected, we can divide by $(1-\gamma)$ and get

$$\gamma \leq \frac{\lambda}{1-\lambda}.$$

The theorem follows. □

4 Summary

There are two important parts of this proof:

1. The Garland method: That
$$\langle Af, g \rangle = \mathop{\mathbb{E}}_{v \in X(0)} [\langle f_v, A_v f_v \rangle].$$
2. The fact that the average part f_v^0 was infact $Af(v)$ (A kills the perpendicular part).

In simplicial complexes, there are other random walks we can decompose to local parts. In the next talk we will talk other random walks that are subject to such a decomposition.

References

[Opp18] Izhar Oppenheim. "Local Spectral Expansion Approach to High Dimensional Expanders Part I: Descent of Spectral Gaps". In: *Discrete Comput. Geom.* 59.2 (2018), pp. 293–330. DOI: 10.1007/s00454-017-9948-x. eprint: 1709.04431.