the  talk

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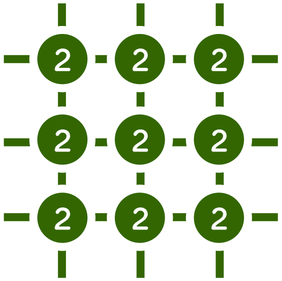
universität wien

2014 | 2 | 24

Simons Institute

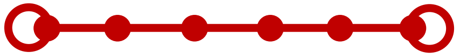
1 strong promises

and eigenvalue gaps



2 the history

of the history state

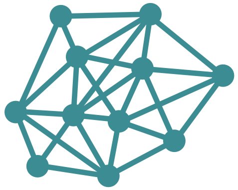


3 running the clock

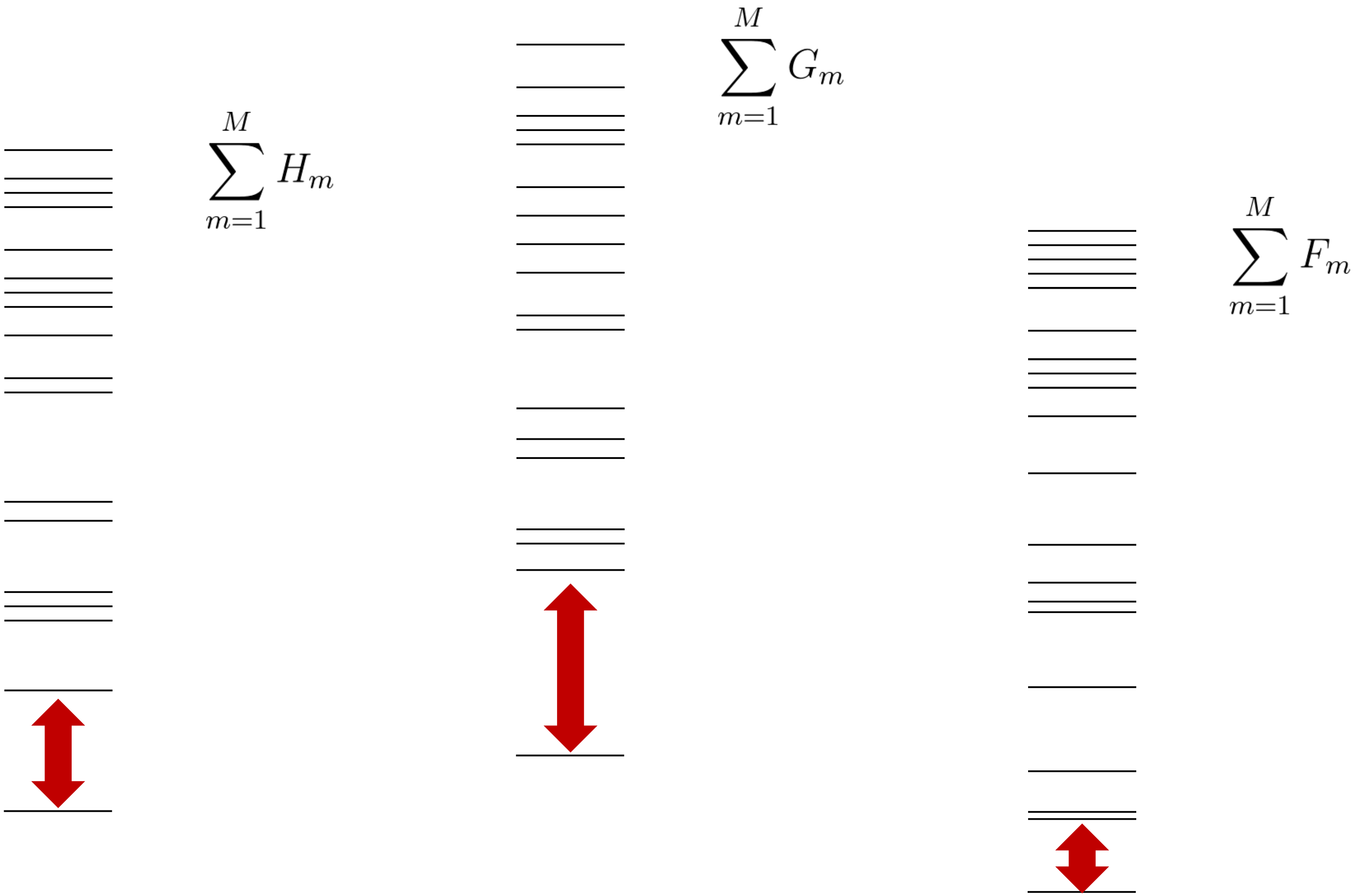
precise/faulty, qubit/qudit, sequential/parallel

4 on the qPCP road

questions & warnings



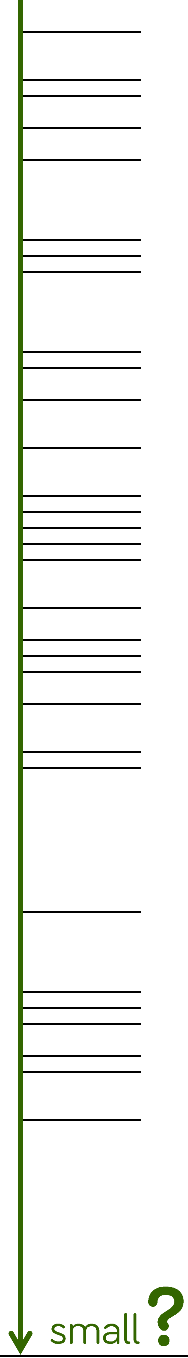
1 Hamiltonians and their eigenvalue gaps

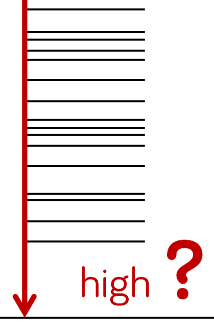


1 Hamiltonians and their ground states

Is
the
ground
state
energy
of a

LH





Is the ground state energy of a

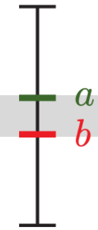
LH

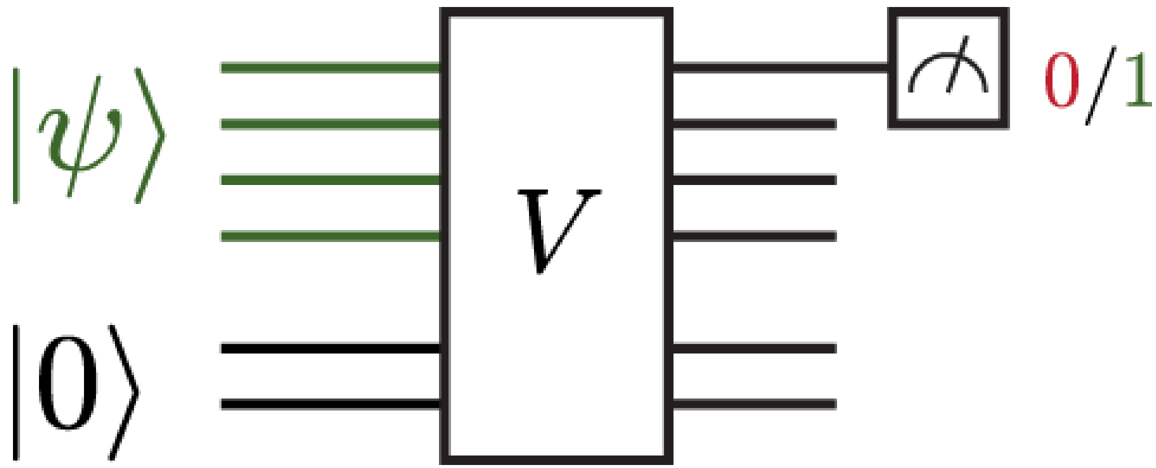
1 The QMA protocol

YES? Accept a good proof with $p > a$.

the promise

NO? Probability of accepting $p < b$.





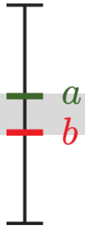
- Is there an acceptable witness for this circuit?
- Is some local Hamiltonian (nearly) frustration-free?

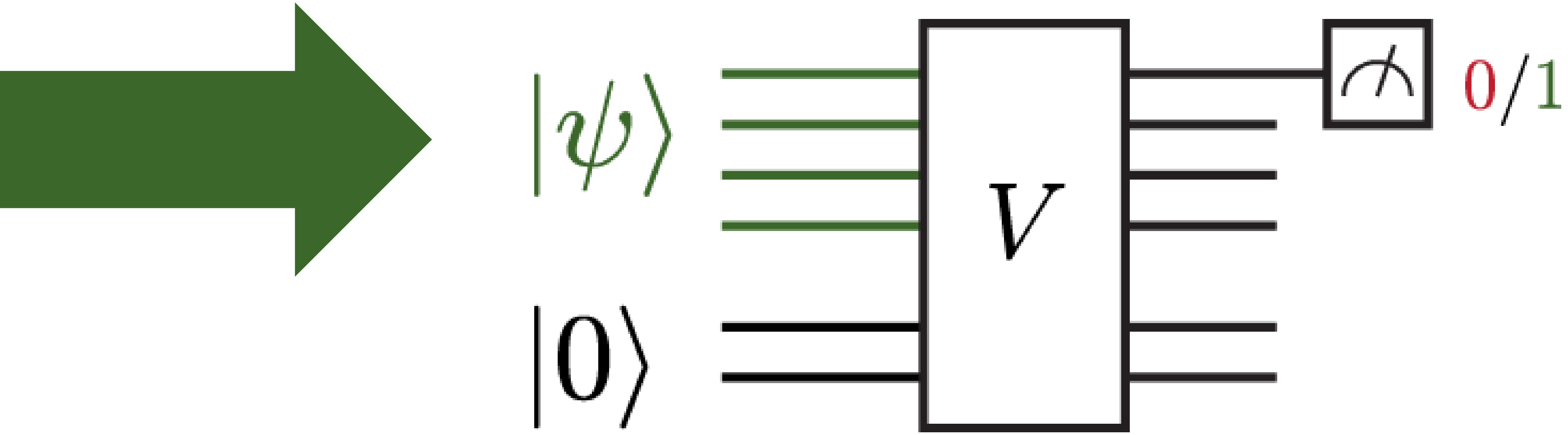
1 The QMA protocol

YES? Accept a good proof with $p > a$.

the promise

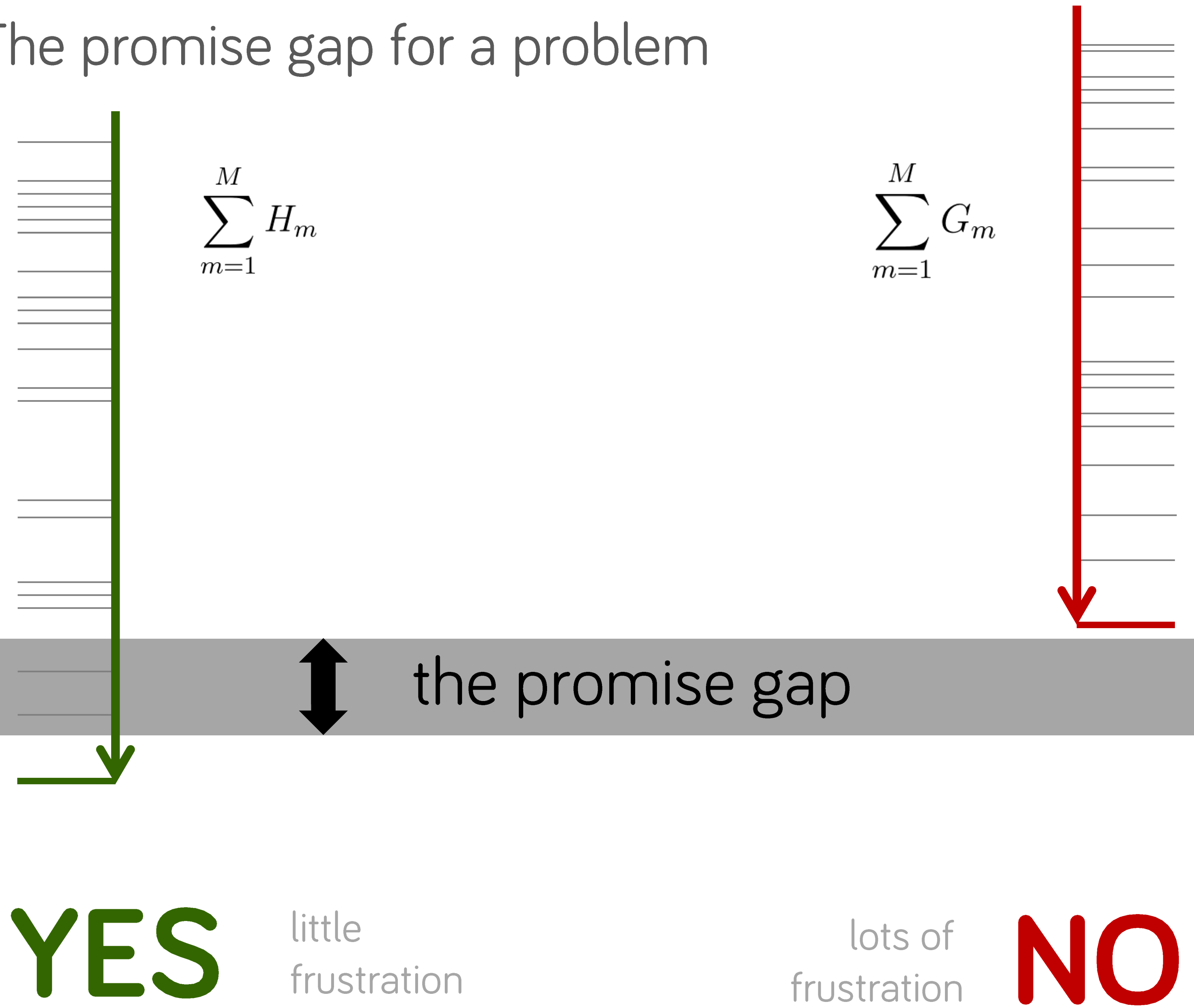
NO? Probability of accepting $p < b$.



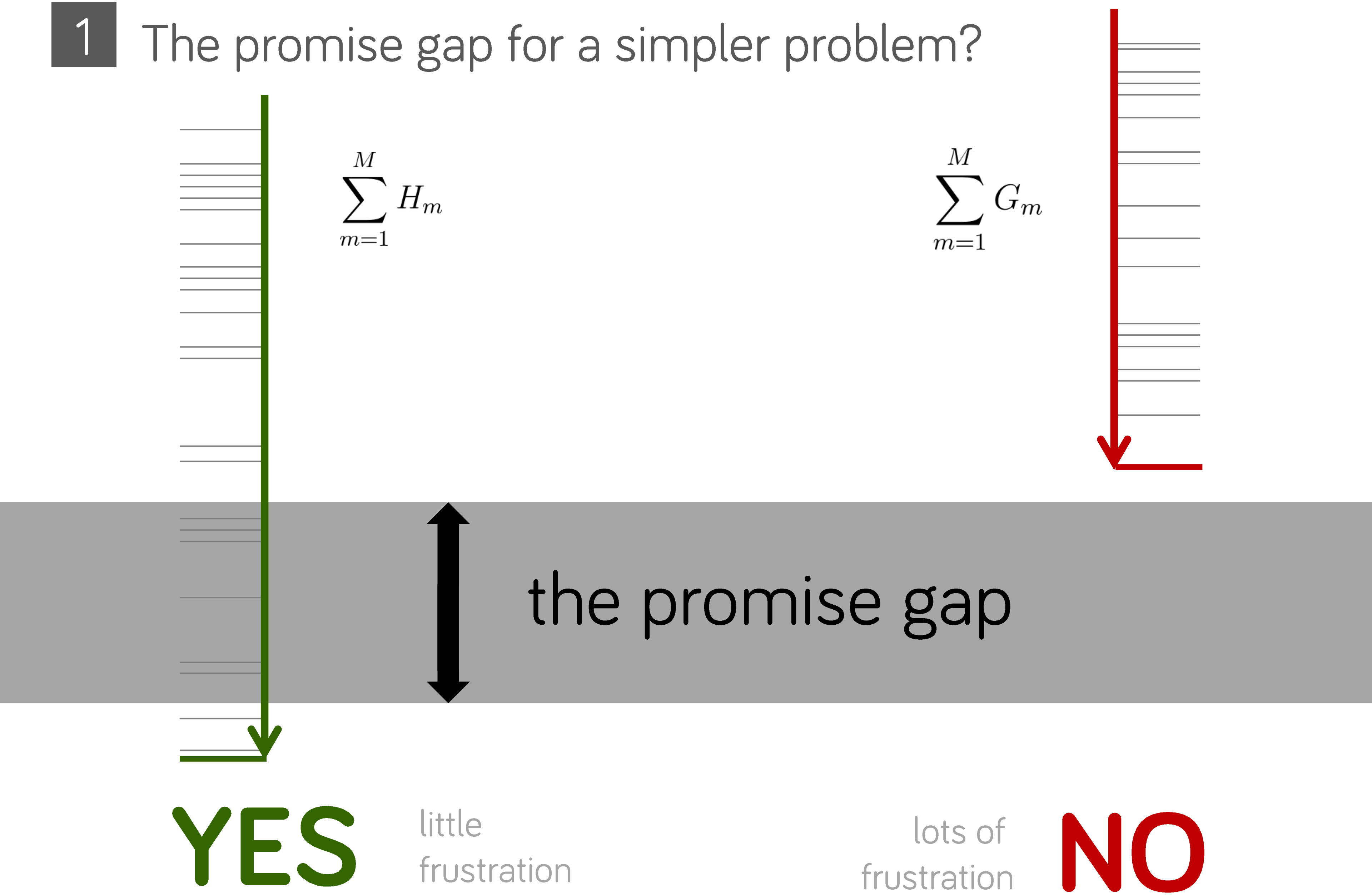


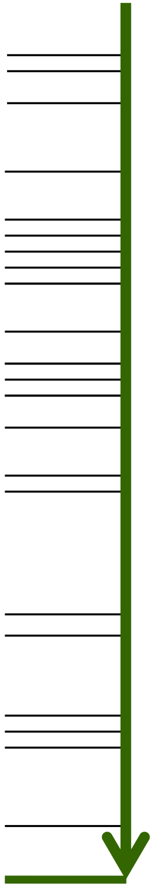
- Is there an acceptable witness for this circuit?
- Does some local Hamiltonian have a low ground energy?

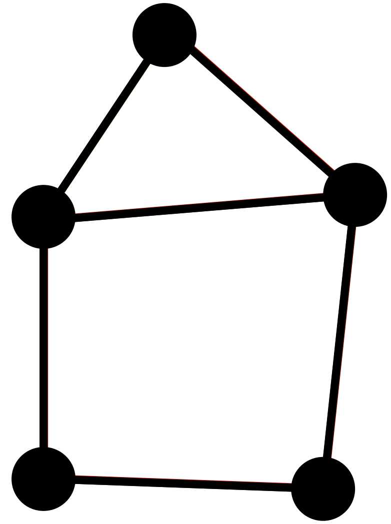
1 The promise gap for a problem

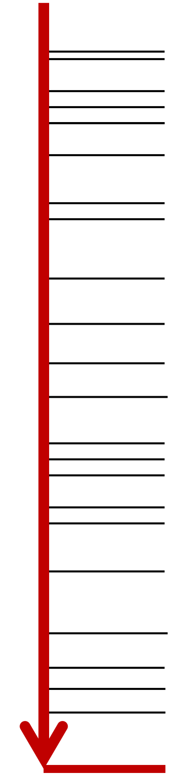


1 The promise gap for a simpler problem?





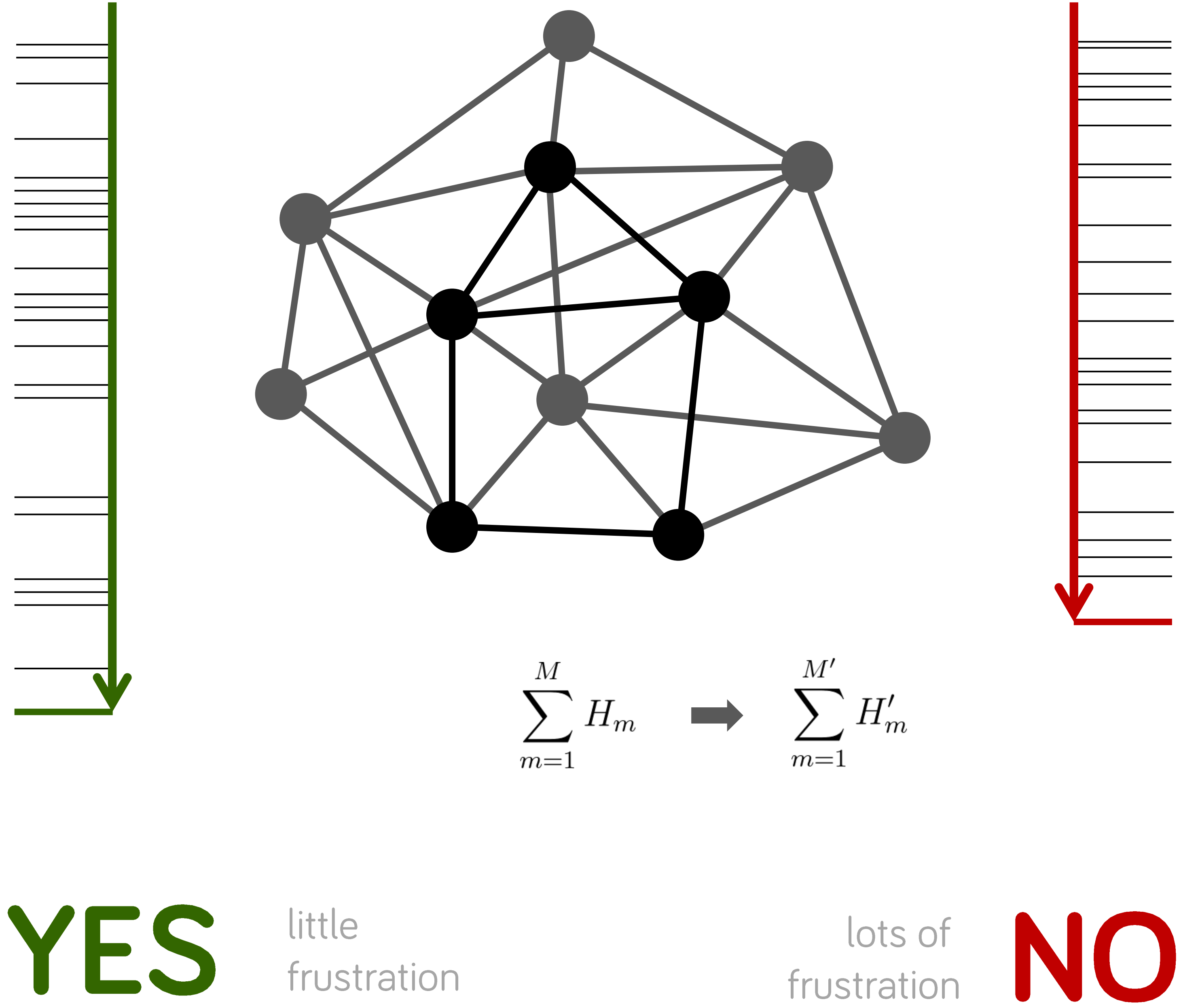




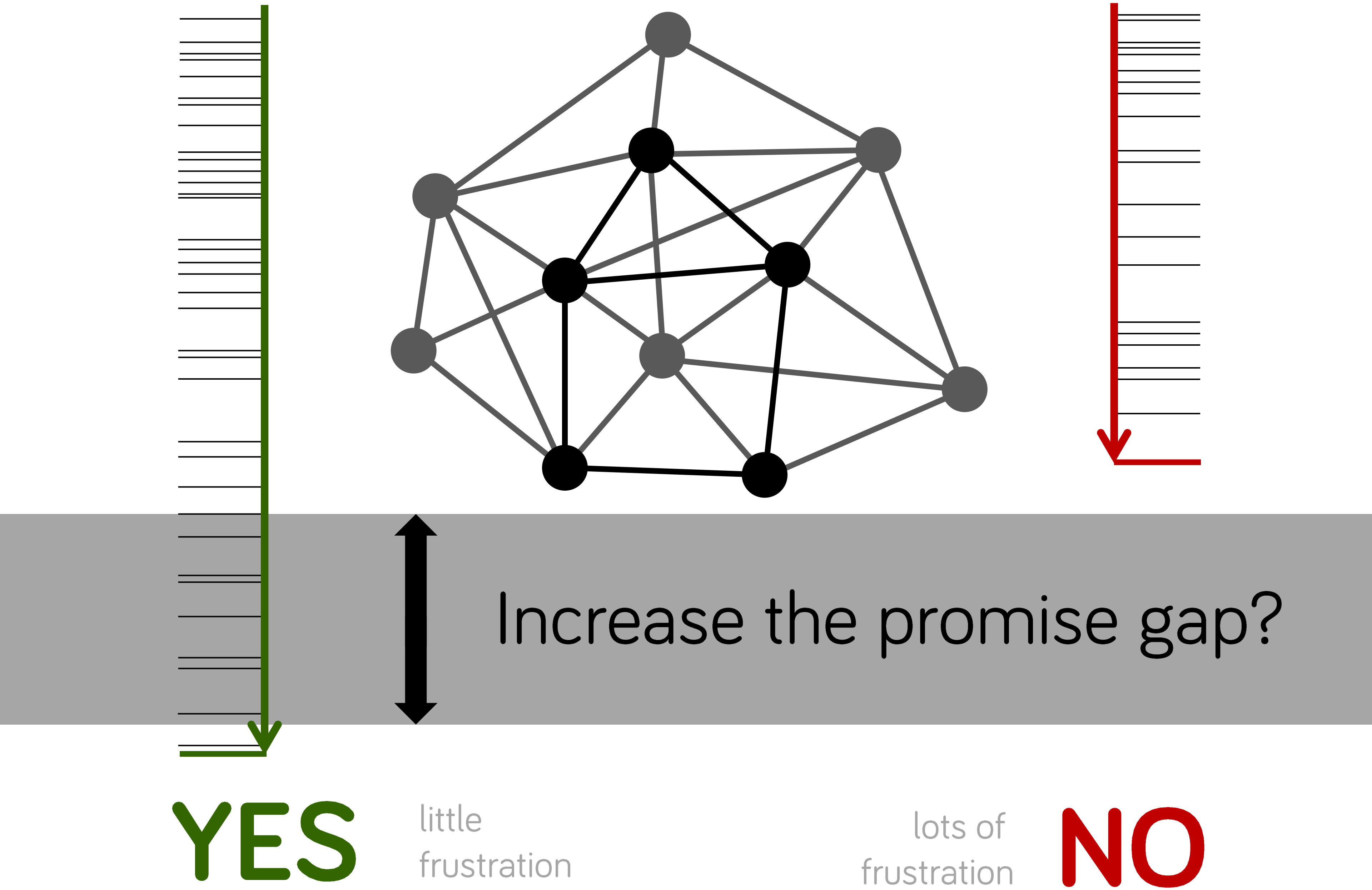
$$\sum_{m=1}^{M} H_m$$

YES little frustration

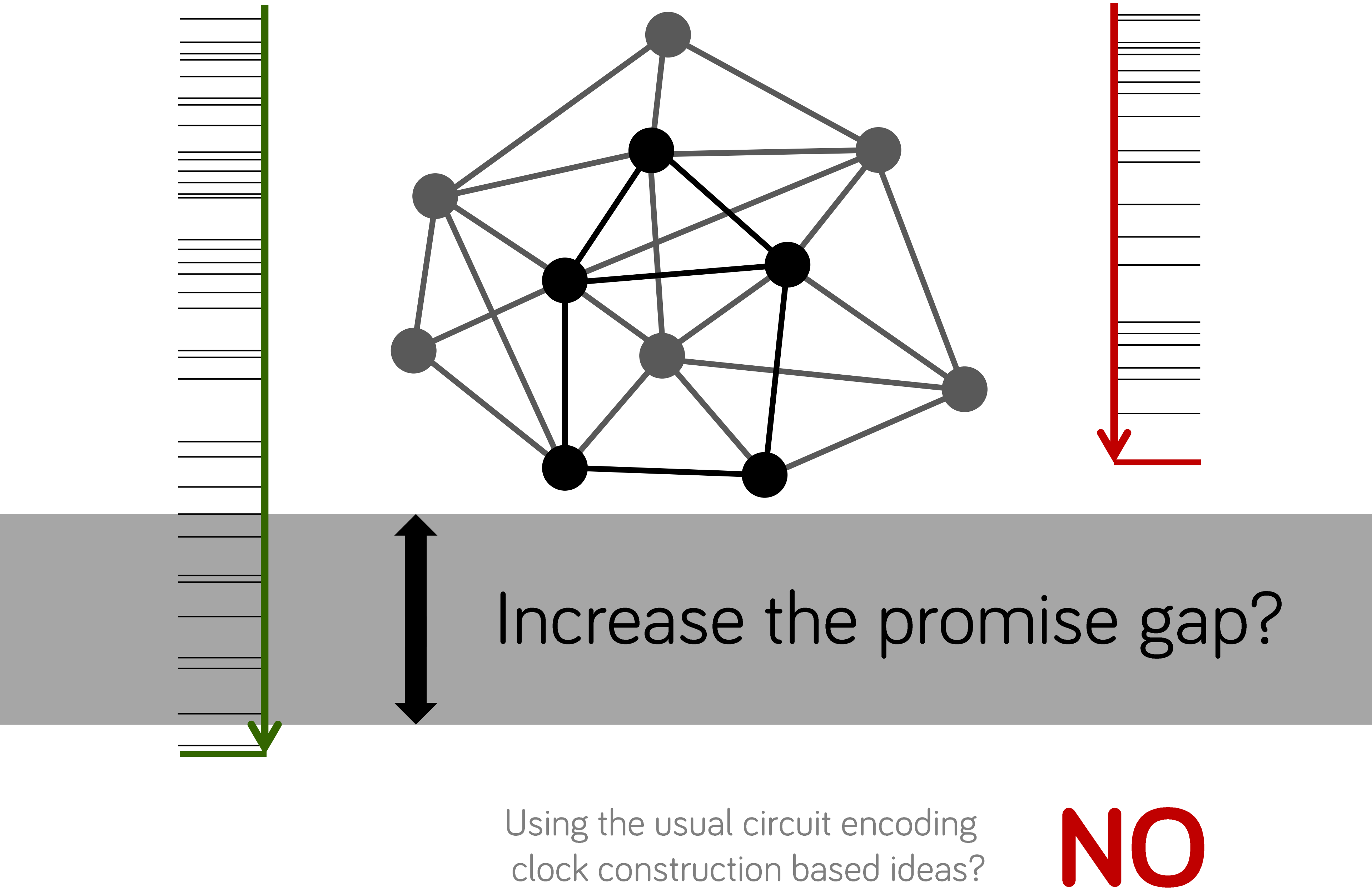
lots of frustration **NO**



$\sum_{m=1}^{M} H_m \rightarrow \sum_{m=1}^{M'} H'_m$
YES
little frustration
lots of frustration
NO



Increase the promise gap?
YES
little frustration
lots of frustration
NO

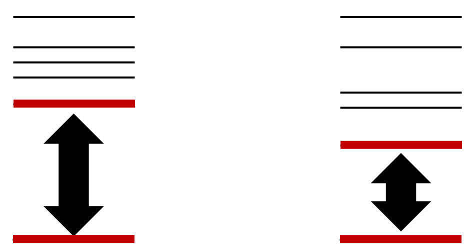

Increase the promise gap?
Using the usual circuit encoding
clock construction based ideas?
NO

1 Small eigenvalue gaps ... small promise gaps

YES little frustration

a very low ground energy

$$H = A + B$$



NO without much frustration

a pretty low ground energy

- geometric lemma

$$\lambda_0 \geq \sin^2 \frac{\vartheta}{2} \times \min(\Delta_A, \Delta_B)$$

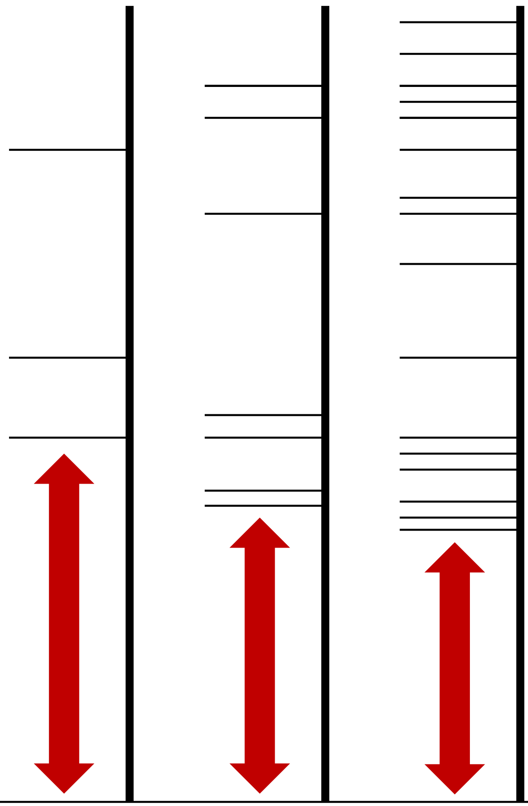
1 Small or large eigenvalue gaps?

$$N \to \infty$$

- Anything close to the ground state?

constant gap? $\Delta \geq const.$

1D: area law, an algorithm
2D: area law?



a Heisenberg XXX spin-1 chain (AKLT)

$$\sum_{j=1}^{N-1} X_j X_{j+1} + Y_j Y_{j+1} + Z_j Z_{j+1}$$

a biased walk in 1D

$$\sum_{j=1}^{N-1} (|j\rangle - B|j+1\rangle)(\langle j| - B\langle j+1|)$$

1 Small or large eigenvalue gaps?

$N \to \infty$

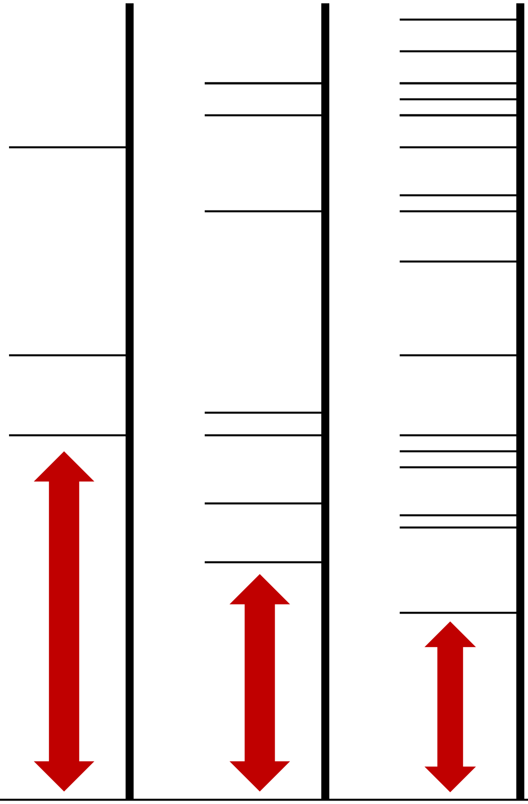
- Anything close to the ground state?

constant gap? $\Delta \geq const.$

1D: area law, an algorithm
2D: area law?

inverse-poly gap? $\Delta \propto N^{-c} \to 0$

clock constructions
NP, QCMA hard
qubits? 1D?



transverse-field Ising

$$\sum_{j=1}^{N-1} X_j - \sum_{j=1}^{N-1} Z_j Z_{j+1}$$

quantum walk on a line

$$\sum_{j=1}^{N-1} |j\rangle\langle j+1| + |j+1\rangle\langle j|$$

1 Small or large eigenvalue gaps? $N \to \infty$

- Anything close to the ground state?

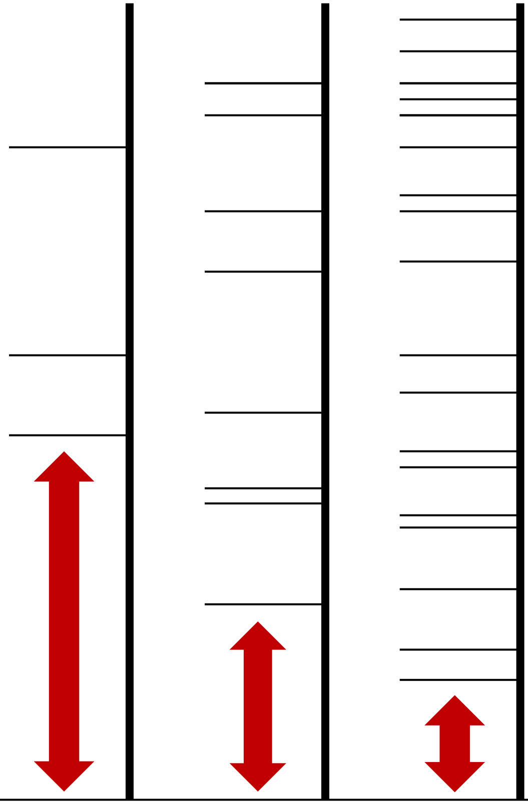
constant gap? $\Delta \geq const.$

1D: area law, an algorithm
2D: area law?

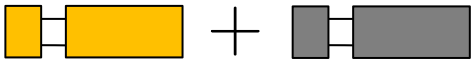
inverse-poly gap? $\Delta \propto N^{-c} \to 0$

clock constructions
NP, QCMA hard
qubits? 1D?

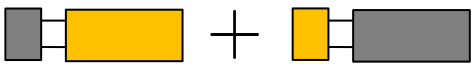
exp-small gap? $\Delta \propto 2^{-cN} \to 0$

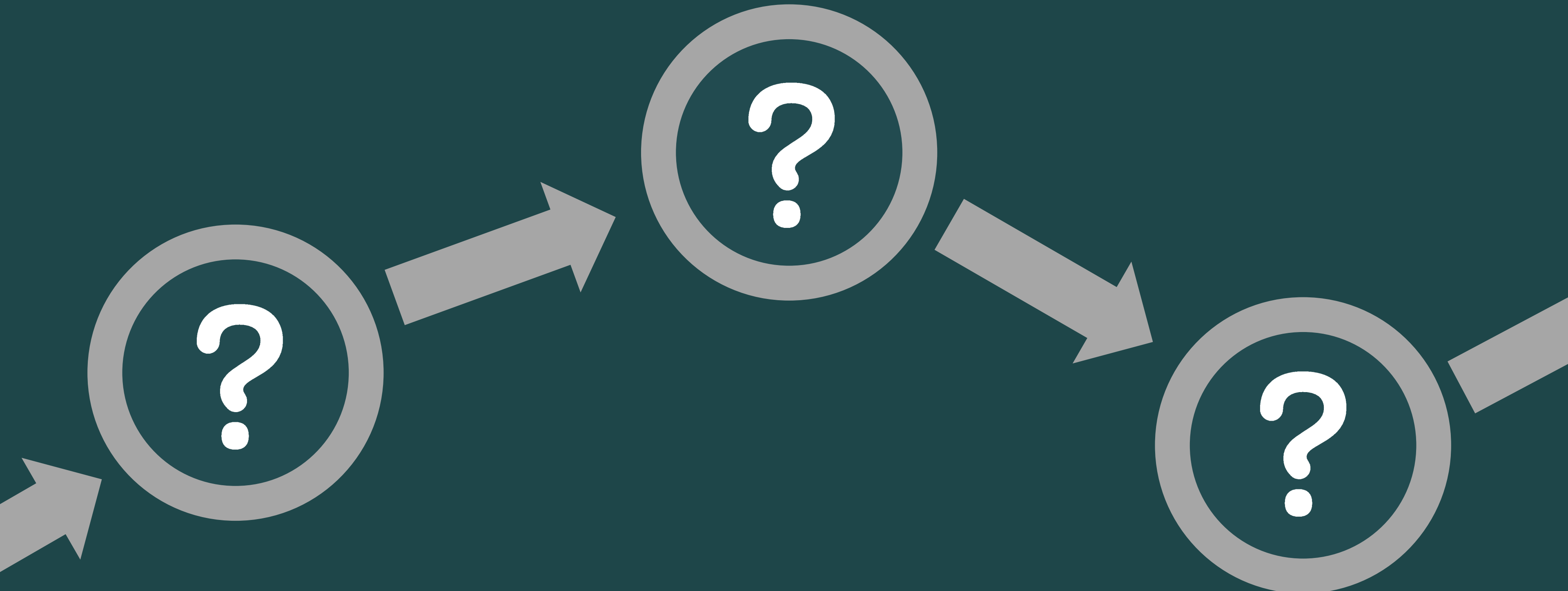


- Constant degree LH: at most constant gap.



- Degeneracy: help or trouble?



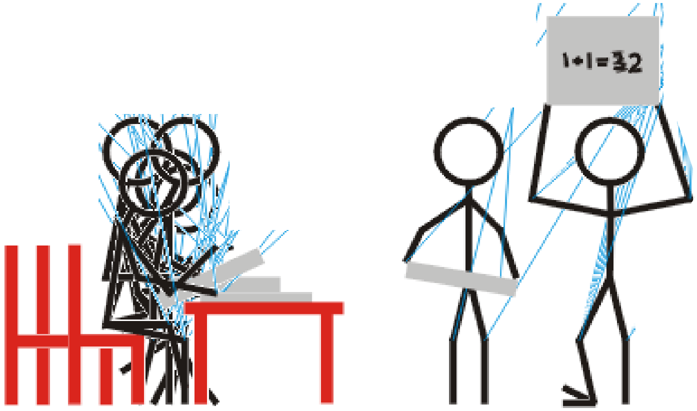


history

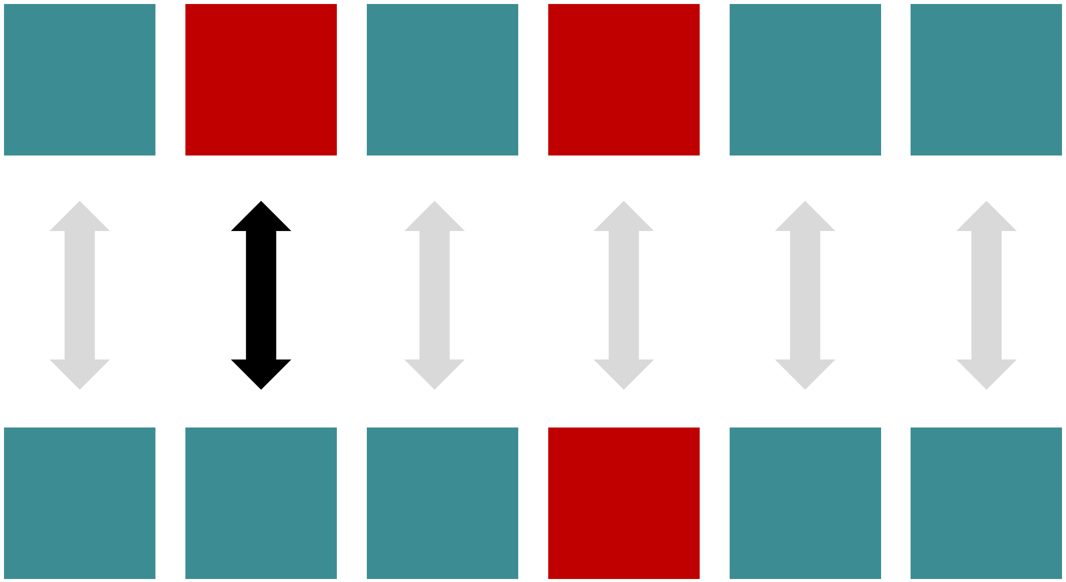
the state

ground

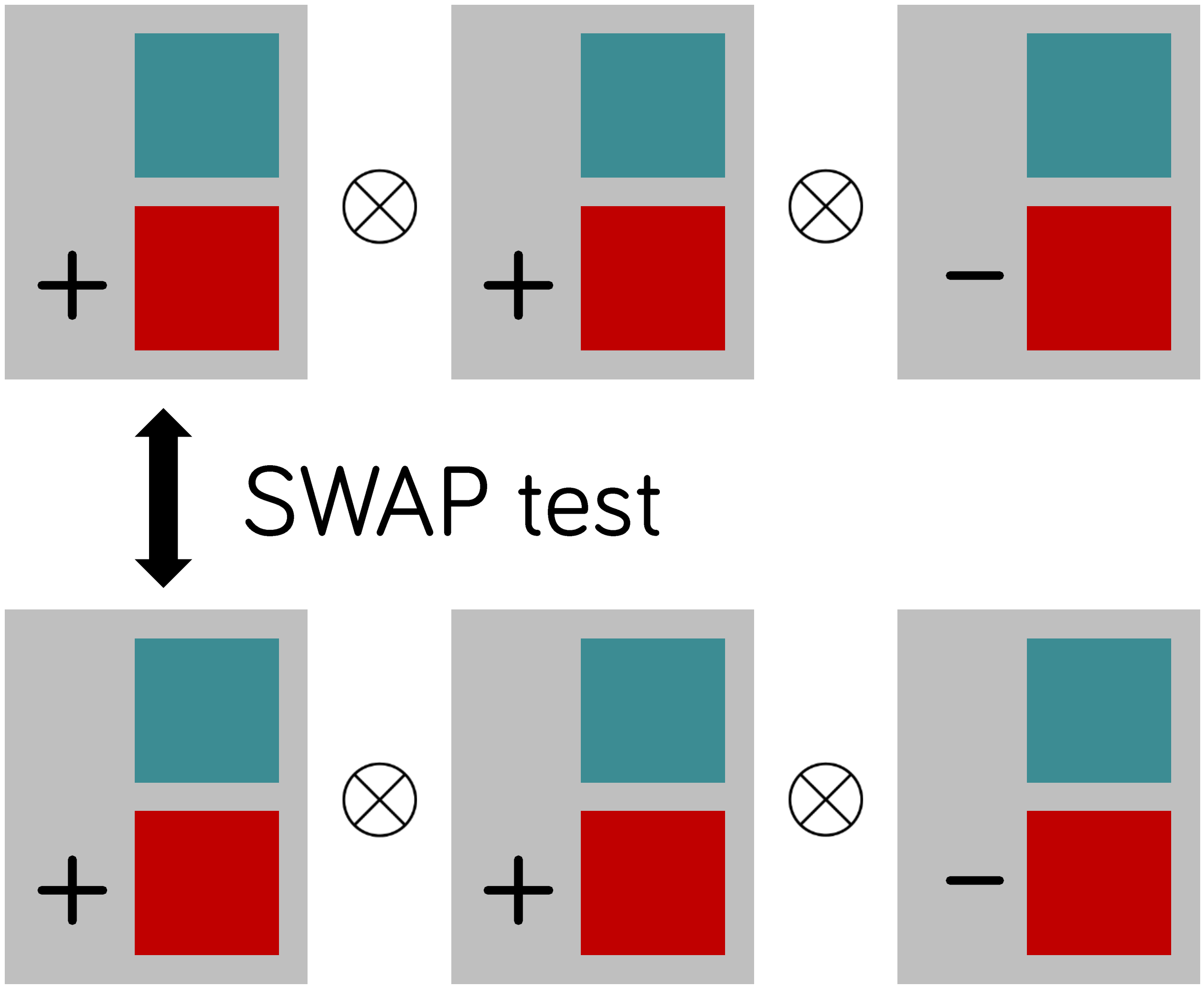
2 Snapshots of a computation



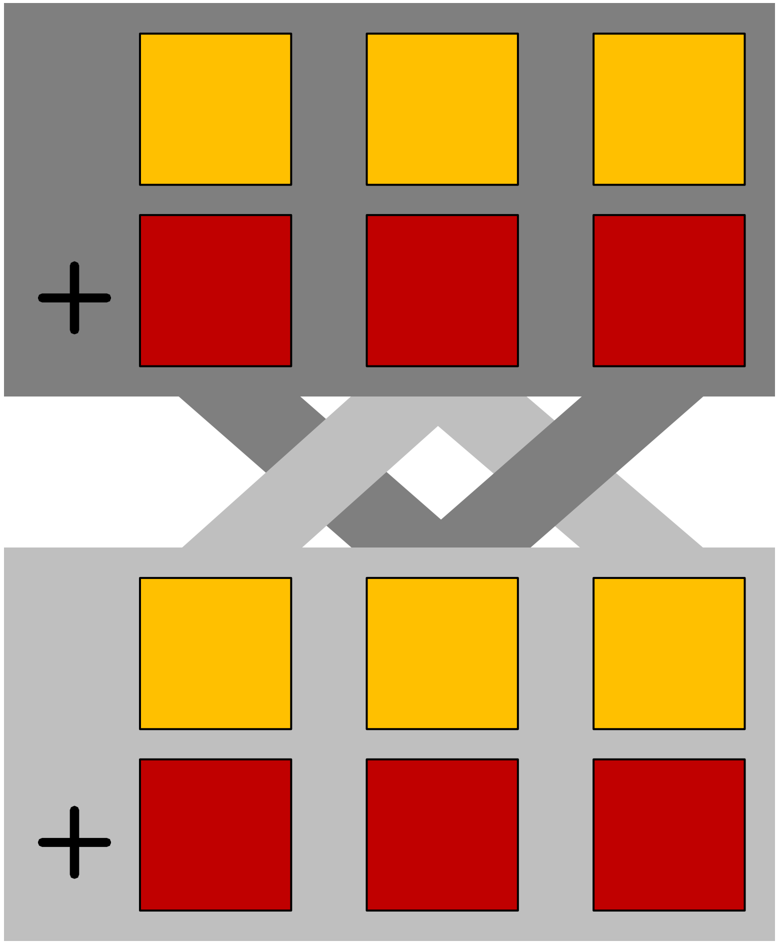
Locally comparing strings.



Locally comparing products.

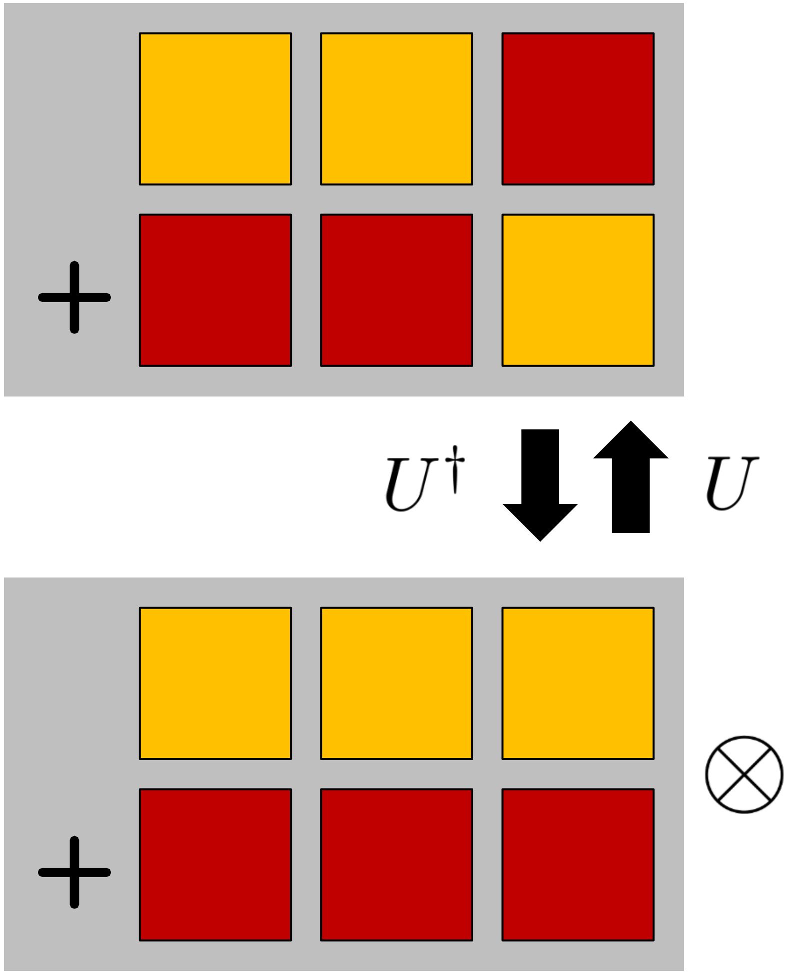


Locally comparing entangled states?



UGH!

2 Labeling the data

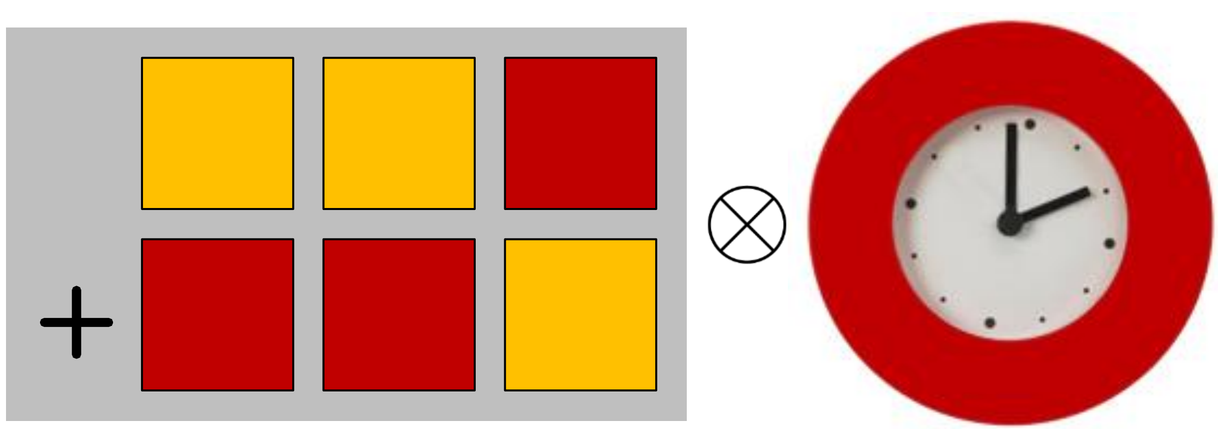


Hard to compare directly (locally).

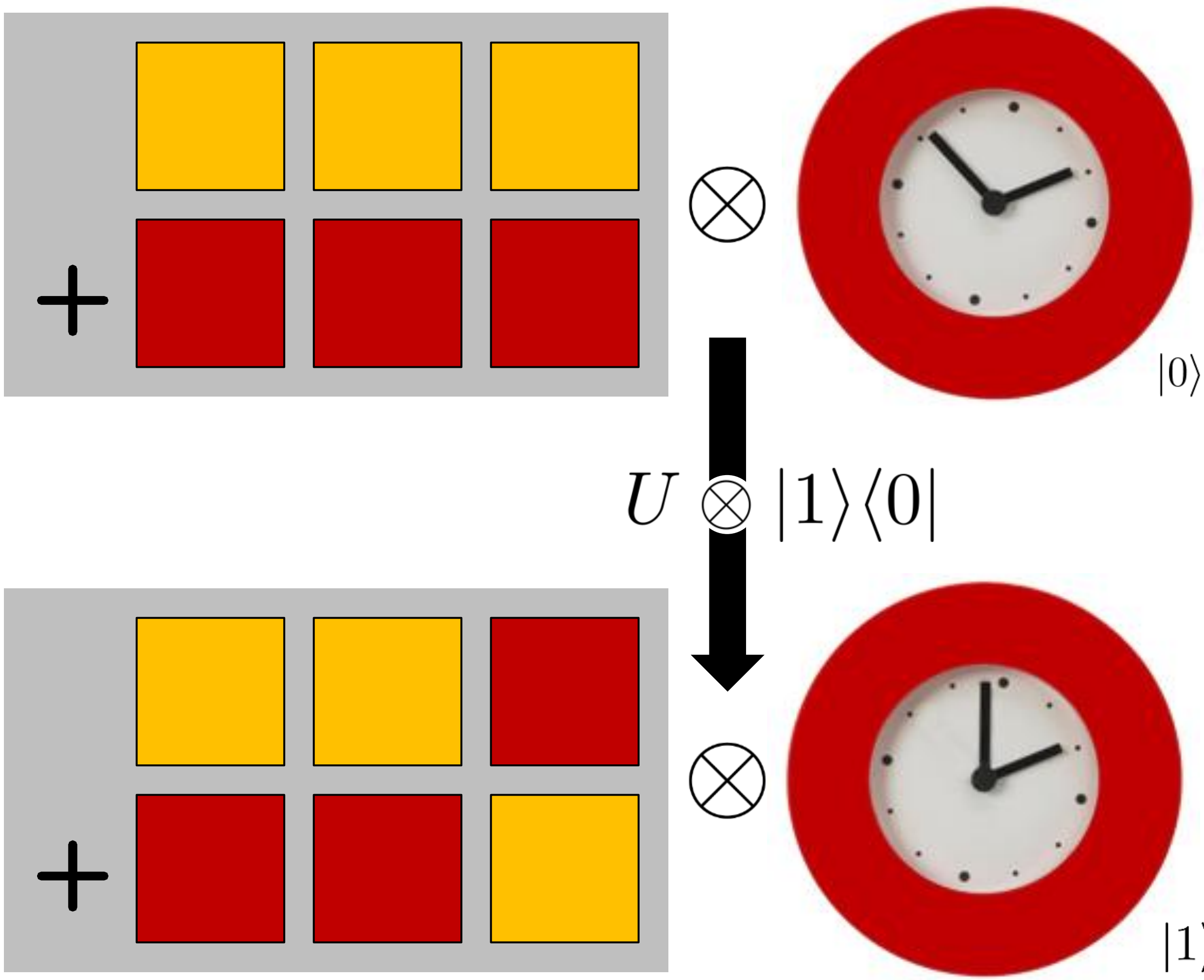
a clock

2 Labeling the data

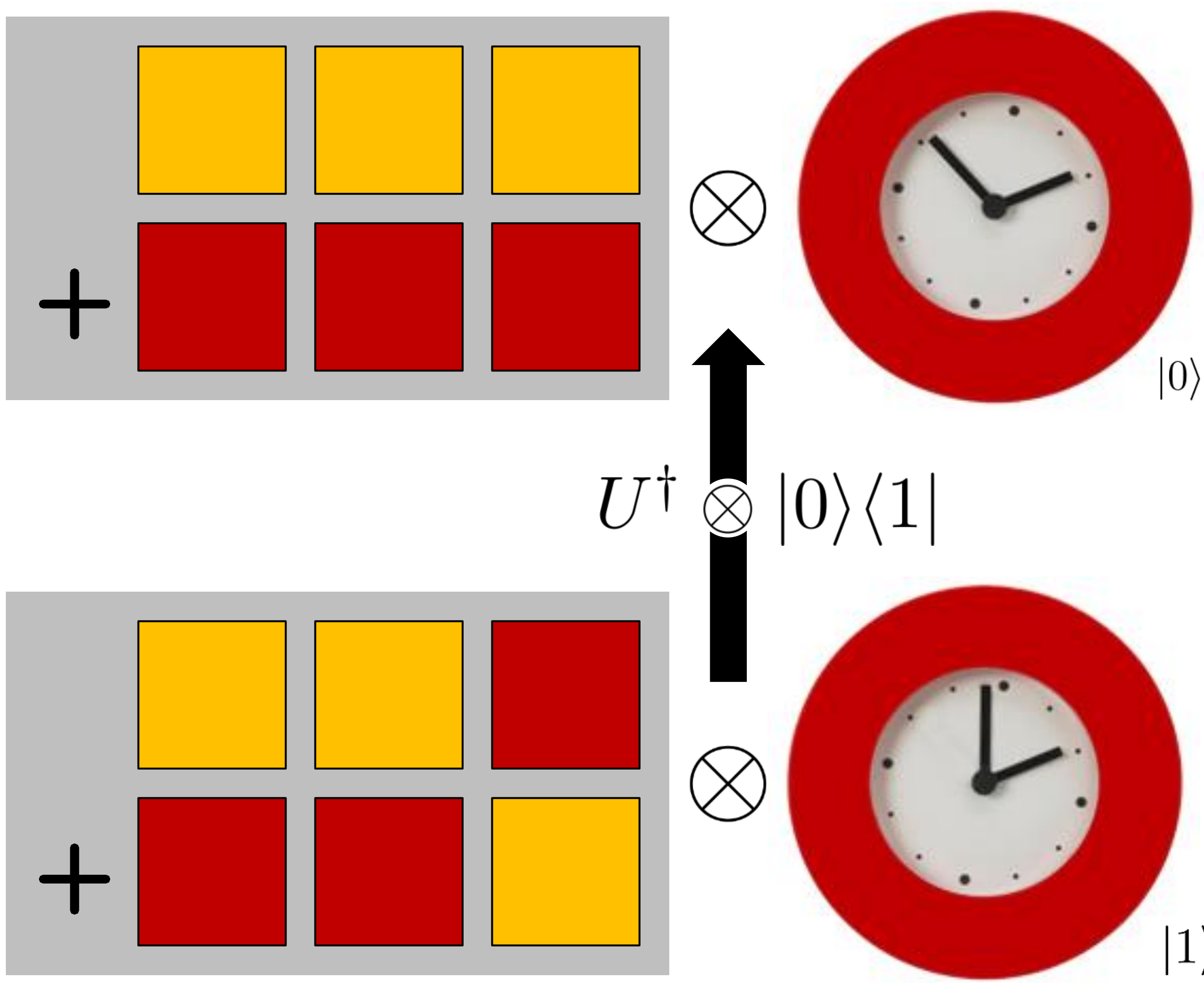
a clock



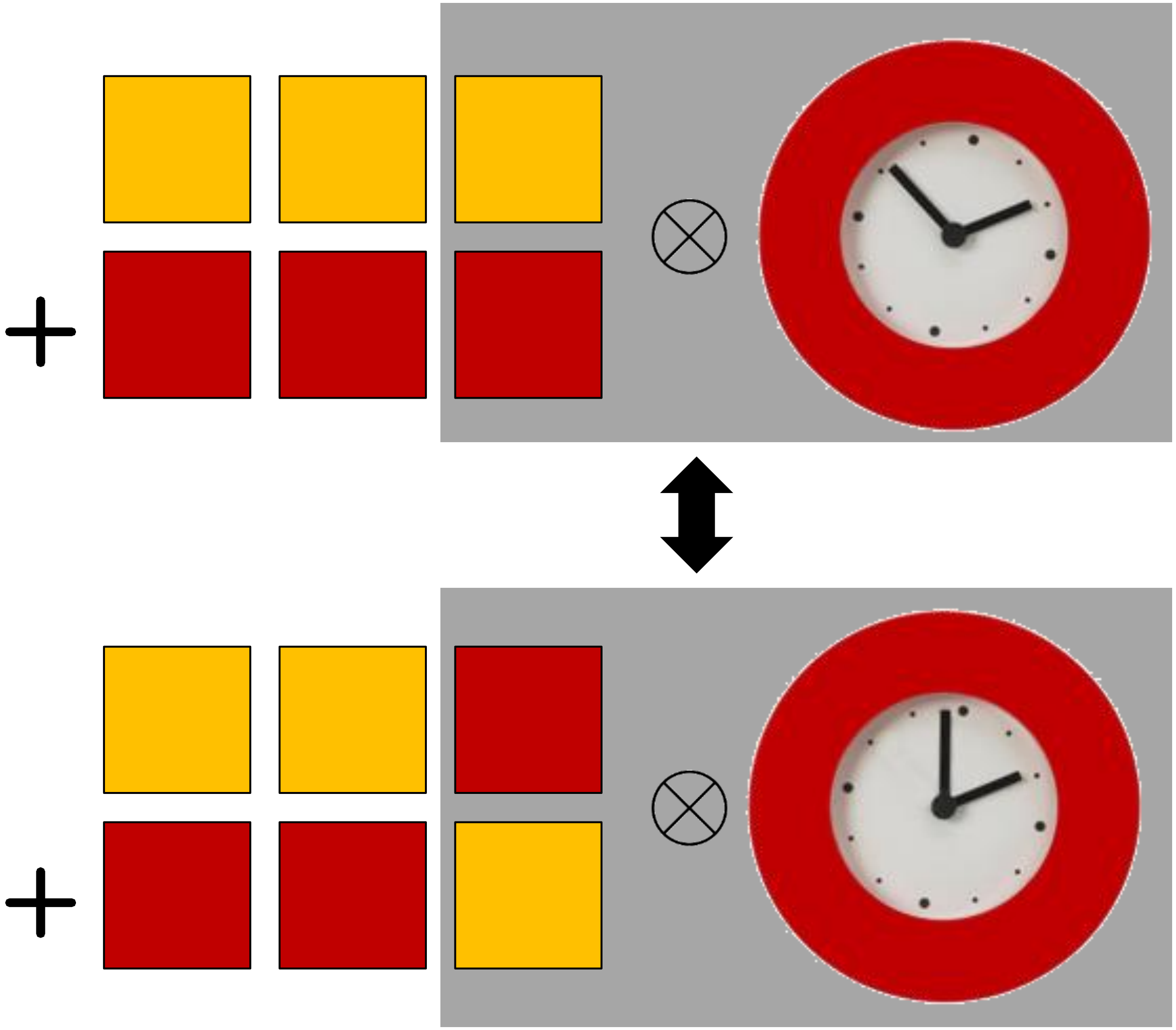
2 The data & the clock



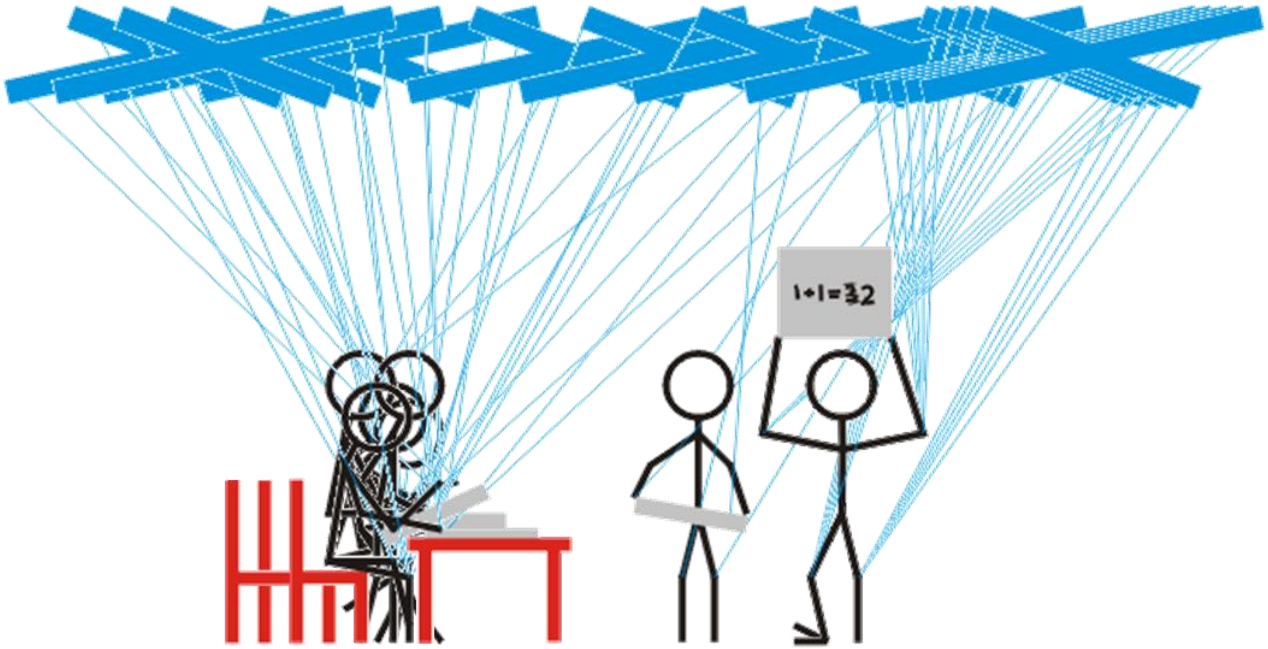
2 The data & the clock



2 The data & the clock: locally comparing related states

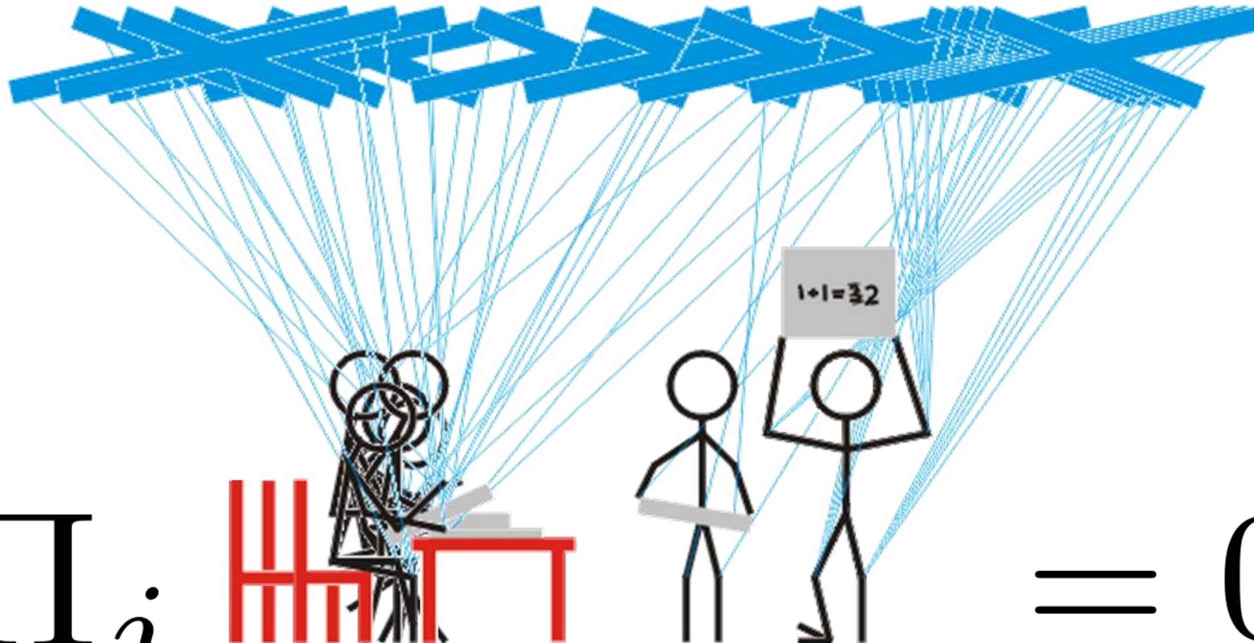


2 The history state



$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} \underbrace{|\varphi_t\rangle}_{U_t \cdots U_1 |\varphi_0\rangle} \otimes |t\rangle$$

2 The history state: a ground state

Π_j  $= 0$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_t\rangle = U_t \cdots U_1 |\varphi_0\rangle$$

k-local

c-o-n-d-i-t-i-o-n-s

clock encoding

state progression

initialization

output $|\cdots 1\rangle \otimes |T\rangle$

$$|\cdots 0\rangle \otimes |0\rangle$$

$$|\varphi_t\rangle \otimes |t\rangle$$

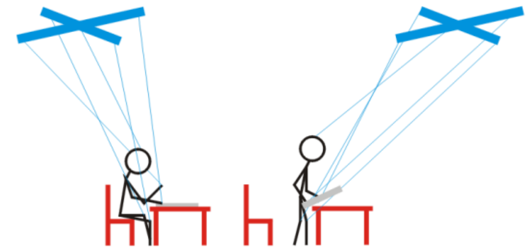
$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

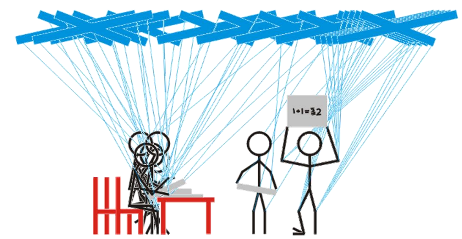
a clock workshop

3 Making a local clock

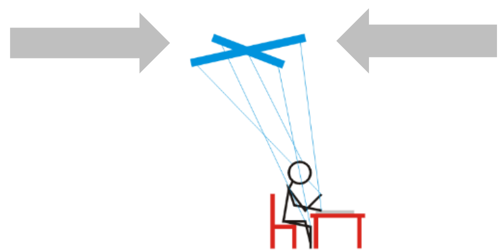
- dynamic: a system that ticks
  local ticks (transitions)



- static: a unique ground state
  the uniform tick superposition



- identifiable states

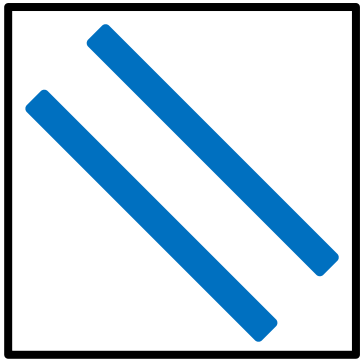


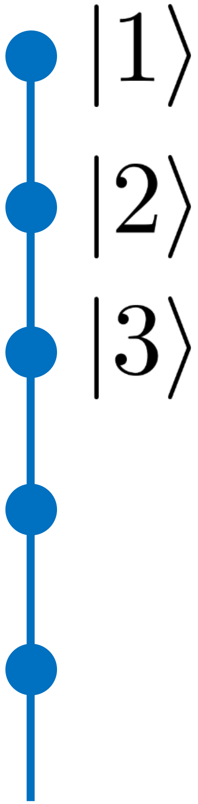
- checking for bad states

3 A quantum walk on a line is a clock

- transitions $T_{st} = |s\rangle\langle t| + |t\rangle\langle s|$

Hamiltonian $H_w = \sum_{\langle s,t\rangle} T_{st}$



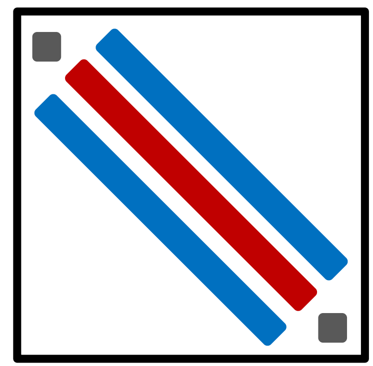


3 A line is a clock

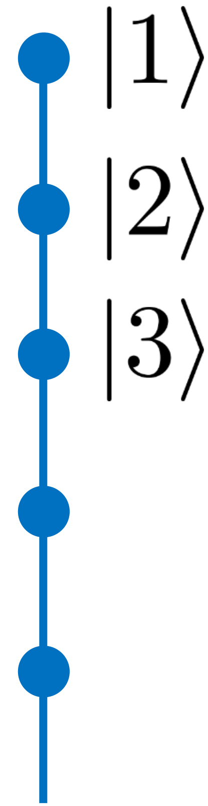
transitions $T_{st} = |s\rangle\langle t| + |t\rangle\langle s|$

- projections $P_{st} = \frac{1}{2}\left(|s\rangle - |t\rangle\right)\left(\langle s| - \langle t|\right)$

Hamiltonian $H = \sum_{\langle s,t\rangle} P_{st}$



the ground state $|1\rangle + |2\rangle + |3\rangle + \cdots$



3 A line is a clock

transitions $$T_{st} = |s\rangle\langle t| + |t\rangle\langle s|$$

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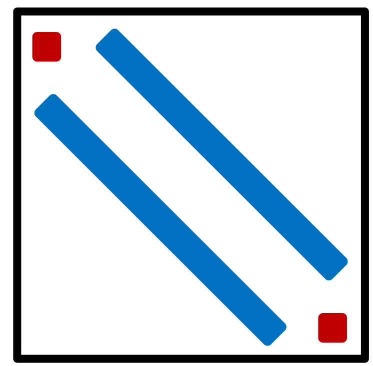
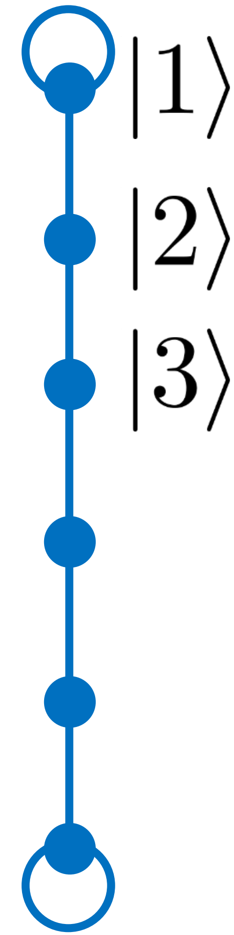
Hamiltonian $$H = \sum_{\langle s,t\rangle} P_{st}$$





the ground state $|1\rangle + |2\rangle + |3\rangle + \cdots$

other eigenstates $$|\varphi_p\rangle \propto \sum_{s=1}^{N} \cos(ps)|s\rangle \qquad E_p = 2\cos p$$

the gap $$\Delta = \Theta\left(\frac{1}{N^2}\right) \qquad p = \frac{k\pi}{N}$$

3 A pulse clock

- transitions
  2-local

$$|10\rangle\langle 01| + |01\rangle\langle 10|$$

- identification

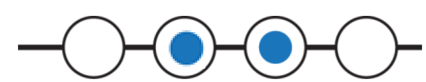
$$|1\rangle\langle 1|$$

- projections
  get a superposition
  for the ground state

$$|01 - 10\rangle\langle 01 - 10|$$

- invariant subspaces & tuning
  a given number of excitations
  tuning for a single excitation: prefer 1, hate 11

1000
+0100
+0010
+0001



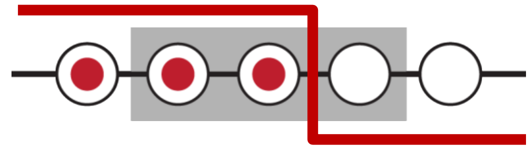
3 A domain wall (unary) clock

- clock checking $|01\rangle\langle 01|$

  2-local

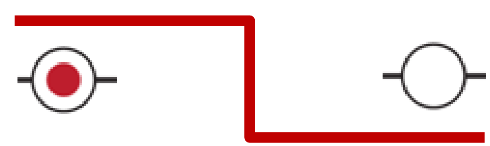
- identification $|10\rangle\langle 10|$

3 A domain wall (unary) clock



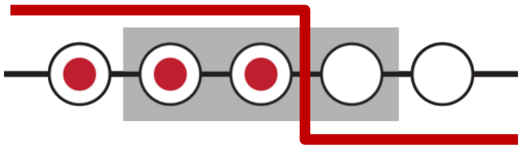
$|3\rangle$

- clock checking $|01\rangle\langle 01|$
  2-local
- identification $|10\rangle\langle 10|$
- projections $|100-110\rangle\langle 100-110|$
  3-local
- a single domain wall: fix the ends
  a unique ground state



10000+11000+11100+11110

3 The DW clock in Kitaev's 5-local Hamiltonian



- clock checking $|01\rangle\langle 01|$

  2-local

- identification $|10\rangle\langle 10|$

- projections $|100 - 110\rangle\langle 100 - 110|$

  3-local

- interacting with data

  5-local

$$\frac{1}{2}\left(|t+1\rangle\langle t+1| + |t\rangle\langle t|\right)$$
$$-\frac{1}{2}\left(U_{t+1}\otimes|t+1\rangle\langle t| + U_{t+1}^{\dagger}\otimes|t\rangle\langle t+1|\right)$$

3 Kitaev's LH: the playground

- Invariant subspaces

  with bad clock states

  labeled by the initial state

$$|\text{stuff}\rangle \otimes |\text{whatever}\rangle$$

$$
\begin{aligned}
&|\varphi_1\rangle \otimes |0\rangle_c \\
U_1\, &|\varphi_1\rangle \otimes |1\rangle_c \\
U_2 U_1\, &|\varphi_1\rangle \otimes |2\rangle_c \\
U_3 U_2 U_1\, &|\varphi_1\rangle \otimes |3\rangle_c \\
U_4 U_3 U_2 U_1\, &|\varphi_1\rangle \otimes |4\rangle_c
\end{aligned}
$$

$$
\begin{aligned}
&|\varphi_1'\rangle \otimes |0\rangle_c \\
U_1\, &|\varphi_1'\rangle \otimes |1\rangle_c \\
U_2 U_1\, &|\varphi_1'\rangle \otimes |2\rangle_c \\
U_3 U_2 U_1\, &|\varphi_1'\rangle \otimes |3\rangle_c \\
U_4 U_3 U_2 U_1\, &|\varphi_1'\rangle \otimes |4\rangle_c
\end{aligned}
$$

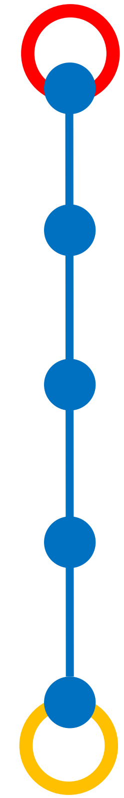
3 The history state: a line of states

- a projector Hamiltonian
  kernel: the uniform superposition

$$
\begin{aligned}
|t+1\rangle\langle t+1| + |t\rangle\langle t| \\
- U_{t+1} \otimes |t+1\rangle\langle t| \\
- U_{t+1}^{\dagger} \otimes |t\rangle\langle t+1|
\end{aligned}
$$

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$



$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

- endpoints: ancilla initialization/final acceptance

- **YES**

ground state

- NO

ground state

- NO

lower bound on the ground state energy

good clock states

not clock states

history states

non-uniform
superpositions

history states

a polynomially small gap

$$\Delta = O\left(L^{-2}\right)$$

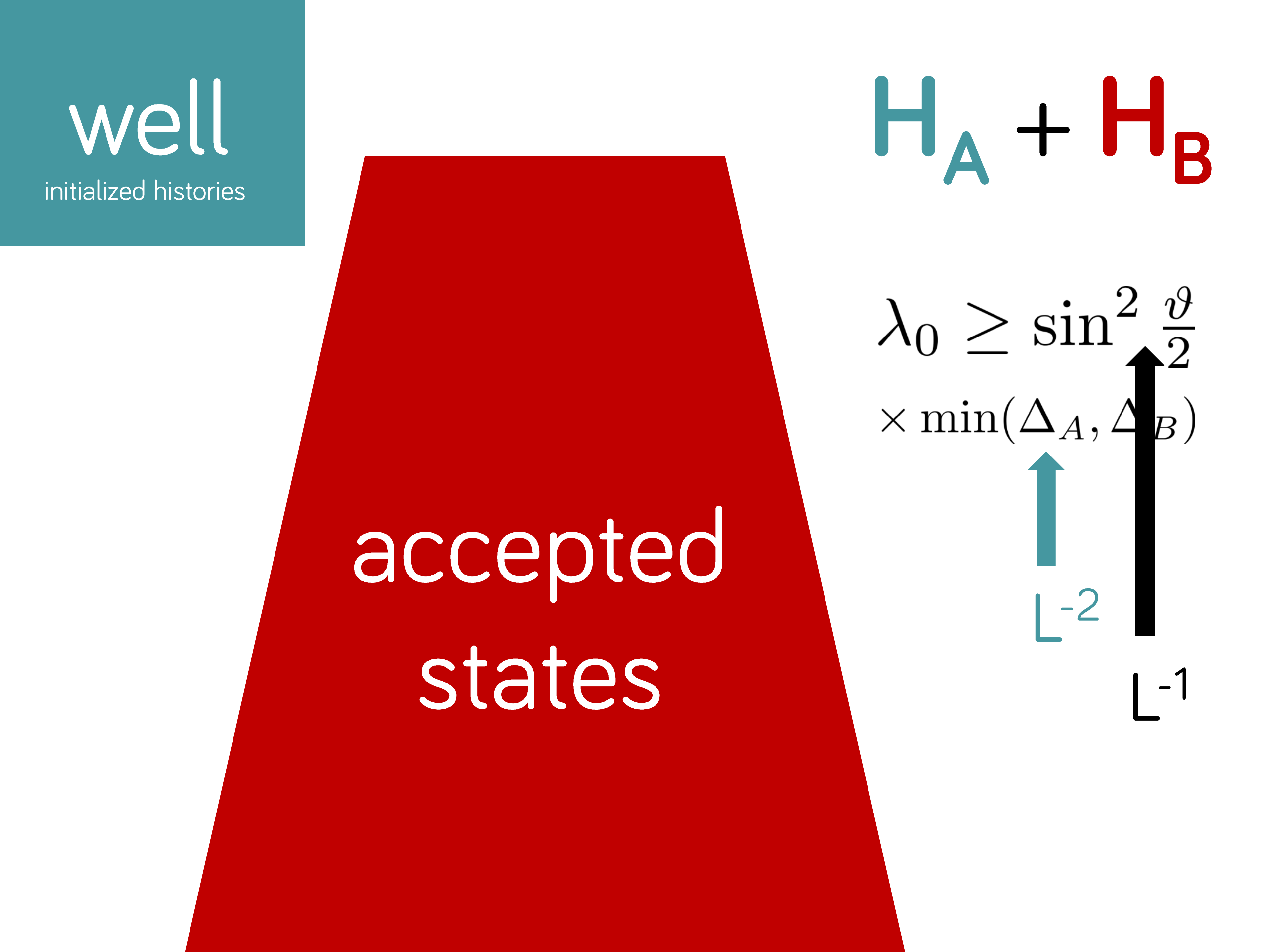
history states

well	badly
initialized history states	

well

initialized histories

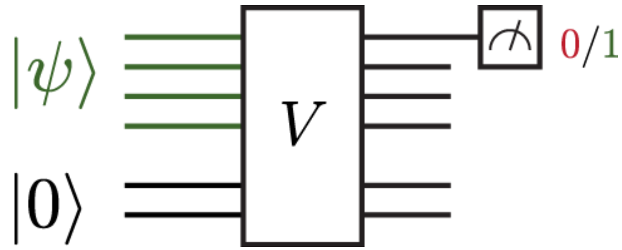
accepted states



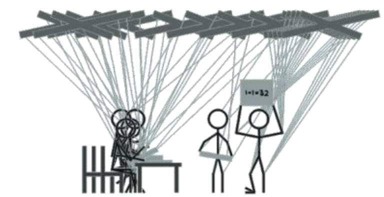
well
initialized histories
accepted
states
$H_A + H_B$
$\lambda_0 \geq \sin^2 \frac{\vartheta}{2}$
$\times \min(\Delta_A, \Delta_B)$
L^{-2}
L^{-1}

3-LH and QMA verification

[N., Mozes 07]



$$H_{clock} + H_{init}$$
$$+ H_{prop} + H_{out}$$



NO V is unlikely to accept anything (ϵ)

lowest eigenvalue

$$\geq \frac{c\left(1-\sqrt{\epsilon}\right)}{L^2}$$

promise gap L^{-2} (needs $\epsilon = L^{-1}$)

YES some proof is likely (1-ϵ) accepted

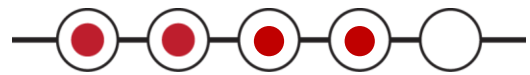
energy of the history

$$\leq \frac{\epsilon}{L+1}$$

projections & gadgets

3 Lower locality for the price of bad transitions

- the domain wall



- projections
  now just 1-local

$$|1-0\rangle\langle 1-0|$$

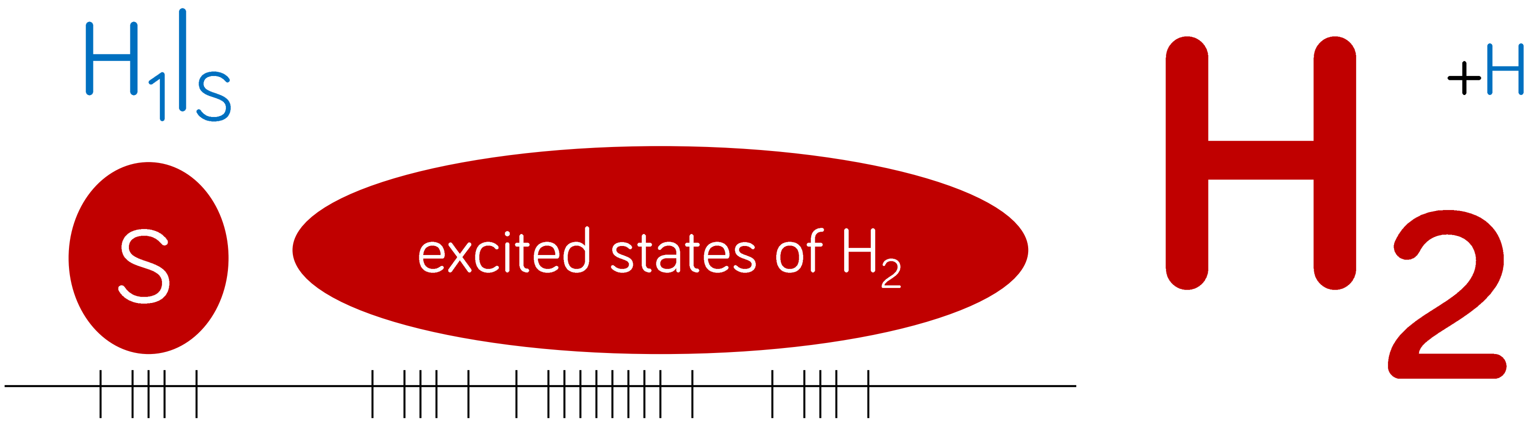
- clock checks
  2-local, STRONG

$$|01\rangle\langle 01|$$

- the ground state is close to

10000+11000+11100+11110

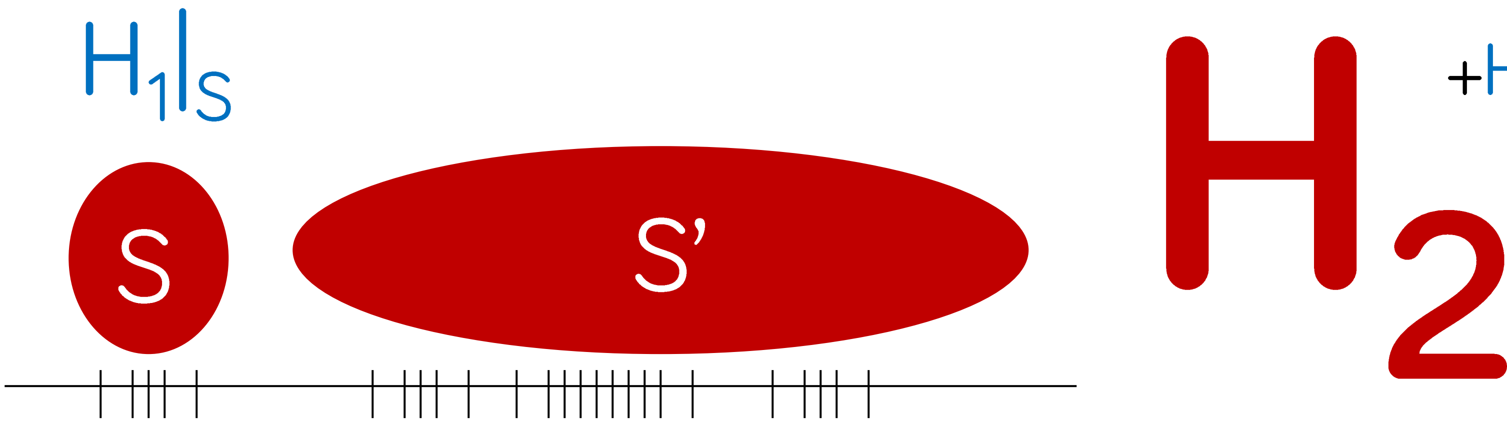
3 The projection lemma to estimate eigenvalues



Lemma 1 *Let* $H = H_1 + H_2$ *be the sum of two Hamiltonians operating on some Hilbert space* $\mathcal{H} = \mathcal{S} + \mathcal{S}^{\perp}$. *The Hamiltonian* H_2 *is such that* $\mathcal{S}$ *is a zero eigenspace and the eigenvectors in* $\mathcal{S}^{\perp}$ *have eigenvalue at least* $J > 2\|H_1\|$. *Then,*

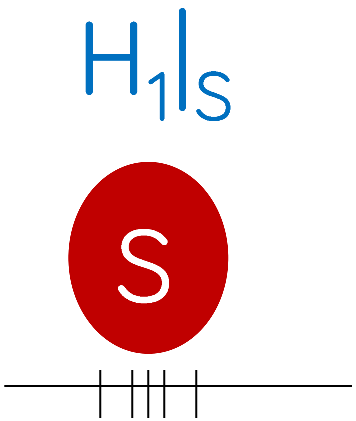
$$\lambda(H_1|_{\mathcal{S}}) - \frac{\|H_1\|^2}{J - 2\|H_1\|} \leq \lambda(H) \leq \lambda(H_1|_{\mathcal{S}})$$

3 The projection lemma to estimate eigenvalues



$$|1-0\rangle\langle 1-0| + |01\rangle\langle 01|$$

3 The projection lemma to estimate eigenvalues



$$|1110000\rangle$$
$$|1111000\rangle$$
$$|1111100\rangle$$

$$|1-0\rangle\langle 1-0| + |01\rangle\langle 01|$$

3 Lower locality (3-LH) for the price of bad transitions

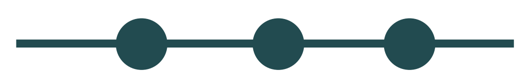
- the domain wall 

- non-projector 2-local terms $$|10\rangle\langle 10|_{1,2} + |10\rangle\langle 10|_{2,3} - X_2$$

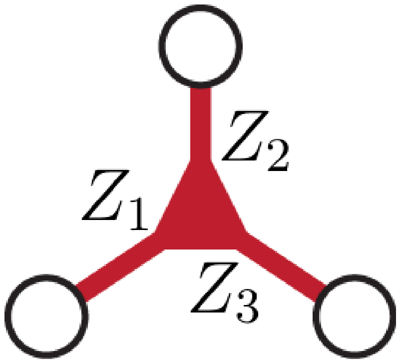
- clock checks 2-local, BIG $$|01\rangle\langle 01|$$

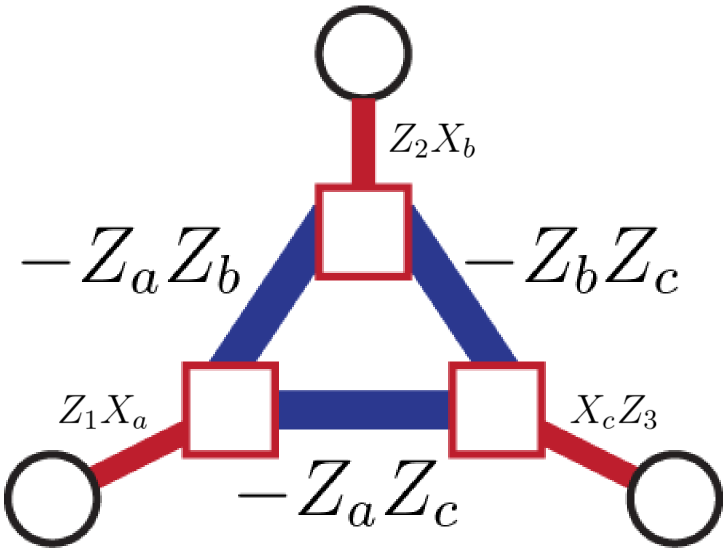
- from 5- to 3-local Hamiltonian [Kempe, Regev]

$$|10\rangle\langle 10|_{1,2} + |10\rangle\langle 10|_{2,3} - |1\rangle\langle 0|_2 \otimes U - |0\rangle\langle 1|_2 \otimes U^\dagger$$

restricted to good clocks: qw on a line 

3 Further decreasing locality: a “3 from 2” gadget





- strongly coupled ancillas (a new energy scale)
- perturbation theory

$$H' = H + V$$

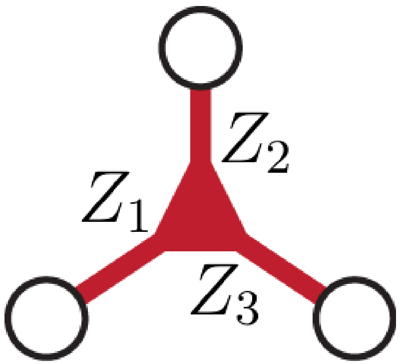
$$||H|| \gg ||V||$$

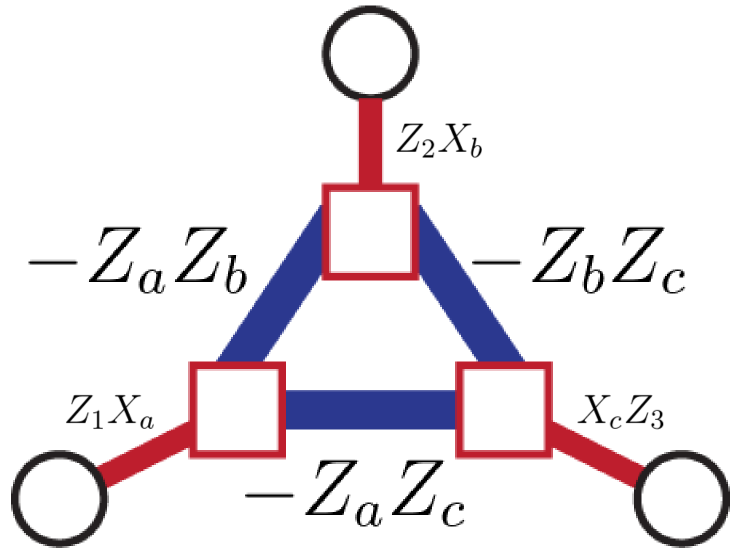
$$G'(z) = \left(z\mathbb{I} - H'\right)^{-1}$$

$$S = \mathrm{span}\,\{|000\rangle, |111\rangle\}$$

[Kempe, Kitaev, Regev ’03]

3 Further decreasing locality: a “3 from 2” gadget





- strongly coupled ancillas (a new energy scale)
- perturbation theory gives us an effective Hamiltonian

$$H' = H + V$$

$$||H|| \gg ||V||$$

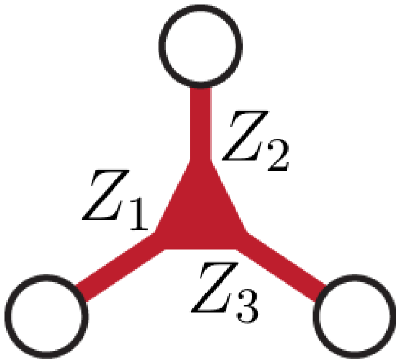
$$S = \mathrm{span}\,\{|000\rangle, |111\rangle\}$$

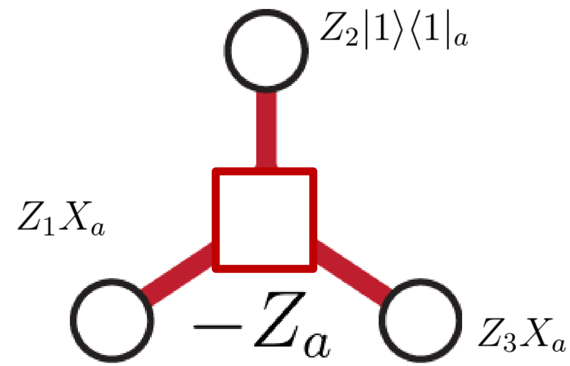
$V\big\|_S$	$V^2\big\|_S$	$V^3\big\|_S$
projection lemma	unwanted (subtract)	the effective 3-local term

[Kempe, Kitaev, Regev '03]

3 STRONG local fields, OK interactions

[Cao et al., 1311.2555]





- strongly bound a single ancilla
  no superstrong interactions

$$S = \{|0\rangle\}$$

- perturbation theory gives us an effective Hamiltonian

$$H' = H + V$$

$$||H|| \gg ||V||$$

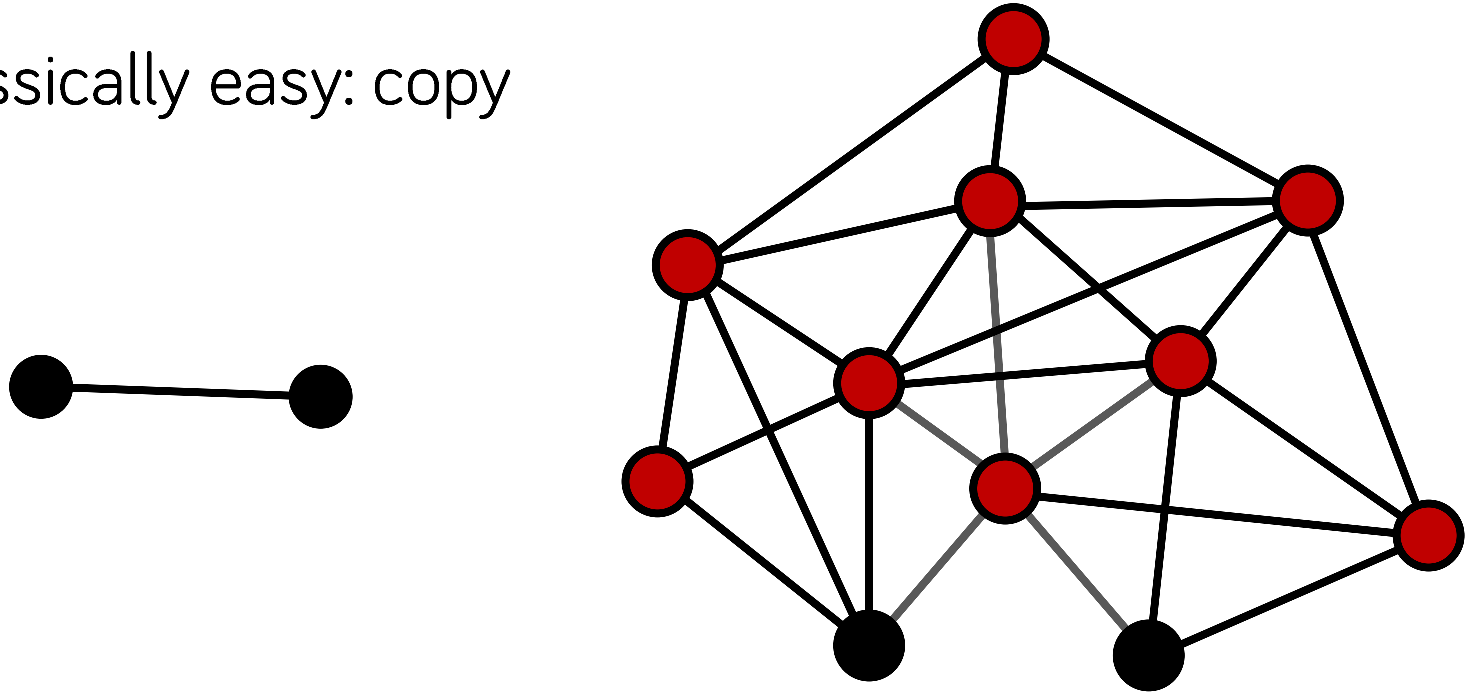
$V\big\|_S$	$V^2\big\|_S$	$V^3\big\|_S$
projection lemma	unwanted (subtract)	the effective 3-local term

special cases (Z-basis)
exact gadgets!

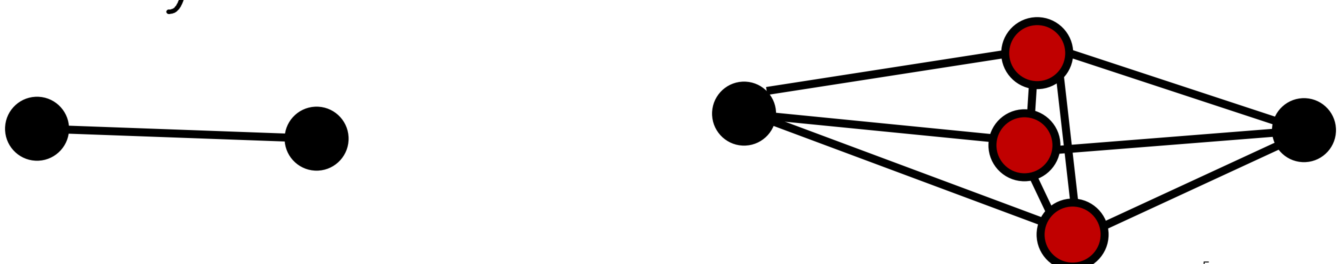
[Jacob Biamonte 0801.3800]

3 "Strengthening", intermediary gadgets?

- classically easy: copy



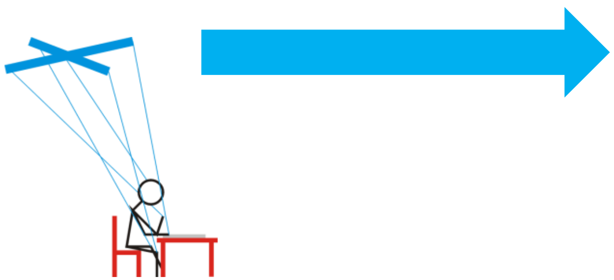
- quantumly?



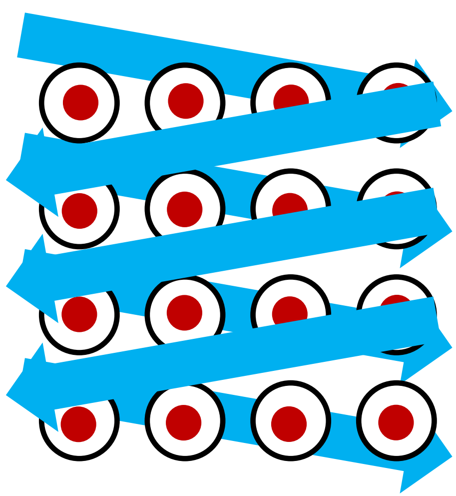
[N., Yudong Cao]

locality
& dimensionality

clock/work registers



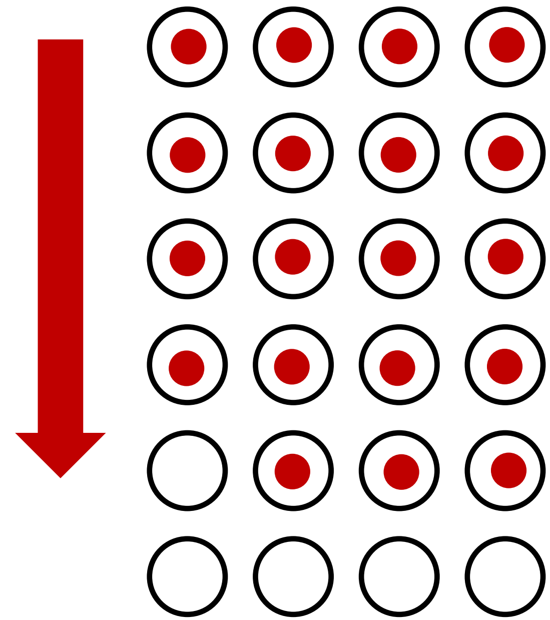
[KKR03, OT05]



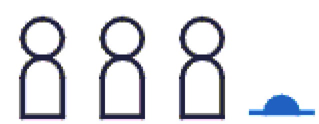
constant degree

geometric locality

a geometric clock

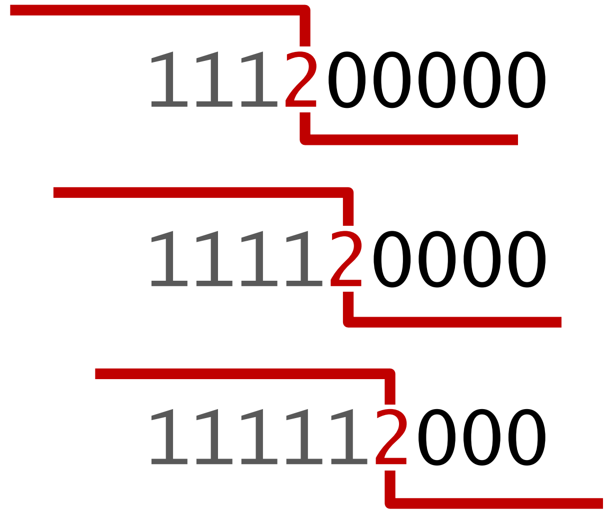


[Mizel] [Janzing] [AvDKLLR] [BT13]



[AGIK07] moving data in 1D

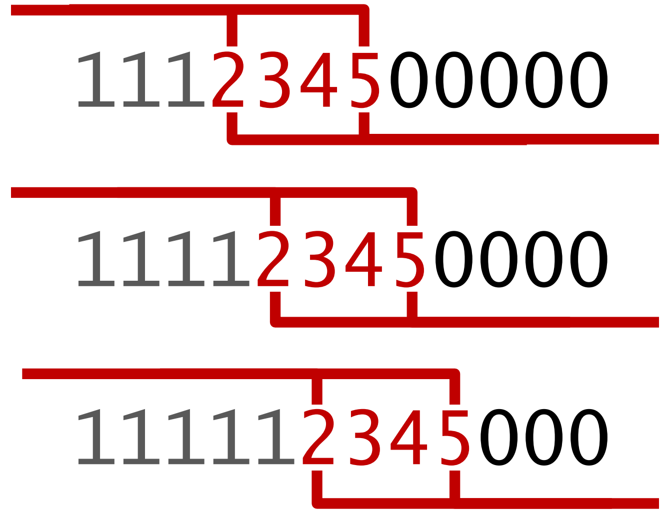
3 Moving a special site: the qutrit surfer



- clock checking $|10\rangle\langle 10|$ $|02\rangle\langle 02|$ $|22\rangle\langle 22|$
  2-local + ends $|01\rangle\langle 01|$ $|21\rangle\langle 21|$
- identification $|2\rangle\langle 2|$
- projections $|20-12\rangle\langle 20-12|$
  2-local

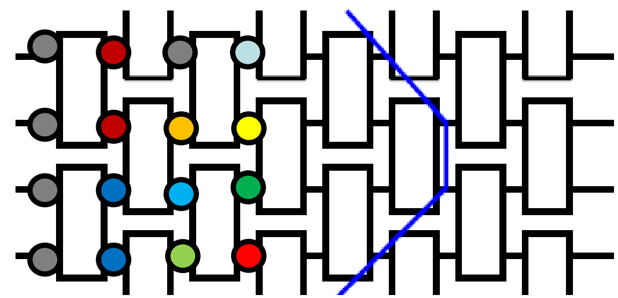
3 Constructing local, geometric clocks: moving the data

- telling "time" by where the data is



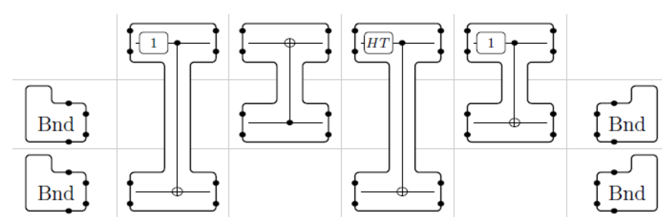
mutually orthogonal states

- carrying/moving data? larger qudits (local dim.) larger locality



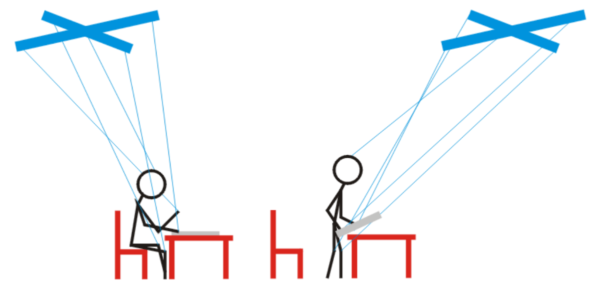
internal states … dual-rail

[Childs Gosset Webb 13]

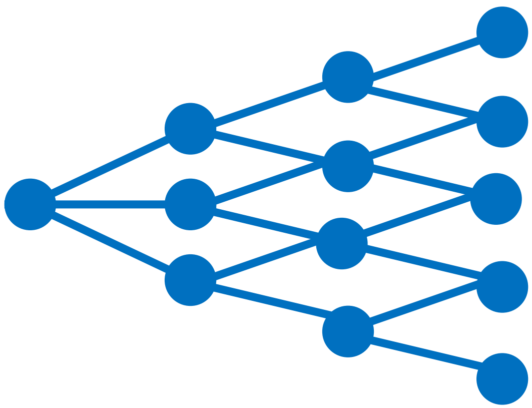


2 Making a good local clock

- identifiable states
  domain-wall structure
  local transitions
  easily checkable bad states



- different geometry?
  simpler terms?
  locality/qudits?
  beyond linear?
  larger (promise) gaps?



2 Hamiltonian Quantum Cellular Automata in 1D

- moving the program instead of the data

[N., Wocjan 07]

program particles diffuse above data
special states stand in their way

2 Hamiltonian Quantum Cellular Automata in 1D

- moving the program instead of the data

[N., Wocjan 07]

program particles diffuse above data
special states stand in their way

2 Hamiltonian Quantum Cellular Automata in 1D

- moving the program instead of the data

[N., Wocjan 07]

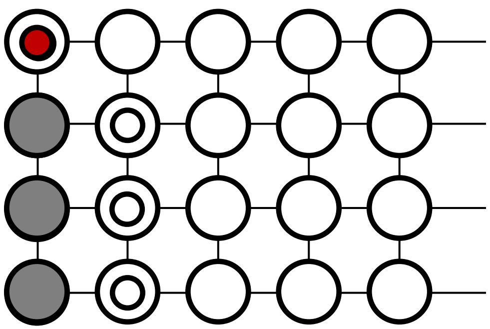
program particles diffuse above data
special states stand in their way

- BQP in 1D with a trans. invariant, time independent LH computational basis programmable
- a nonlinear clock, polynomial expected runtime

3 A local, sequential geometric clock in 2D

- 2D "sequential" evaluation

  [Aharonov van Dam Kempe Landau Lloyd Regev 04]



Forbidden	Guarantees that
○⦶, ○⦷, ○⊗	○ is to the right of all other qubits
○⊗, ⦶⊗, ⦷⊗	⊗ is to the left of all other qubits
○⊗, ⊗○	○ and ⊗ are not horizontally adjacent
⦶⦶, ⦶⦷, ⦷⦶, ⦷⦷	only one of ⦶, ⦷ per row
○ ⦶ ⊗ ⦷, ⦷, ⦷	only ⦷ above ⦷
⦶ ⦶ ⦶ ○, ⦷, ⊗	only ⦶ below ⦶
○ ⊗ ⊗, ○	○ and ⊗ are not vertically adjacent
⦷ ⊗ ○, ⦶	no ○ below ⦷ and no ⦶ below ⊗

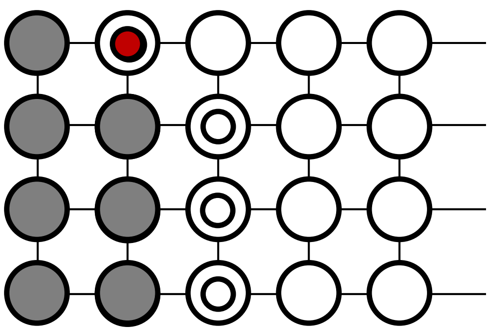
universality of adiabatic QC

2-local interactions, d=6 qudits

3 A local, sequential geometric clock in 2D

- 2D "sequential" evaluation

  [Aharonov van Dam Kempe Landau Lloyd Regev 04]



Forbidden	Guarantees that
○⦶, ○⦷, ○⊗	○ is to the right of all other qubits
○⊗, ⦶⊗, ⦷⊗	⊗ is to the left of all other qubits
○⊗, ⊗○	○ and ⊗ are not horizontally adjacent
⦶⦶, ⦶⦷, ⦷⦶, ⦷⦷	only one of ⦶, ⦷ per row
○ ⦶ ⊗ ⦷, ⦷, ⦷	only ⦷ above ⦷
⦶ ⦶ ⦶ ○, ⦷, ⊗	only ⦶ below ⦶
○ ⊗ ⊗, ○	○ and ⊗ are not vertically adjacent
⦷ ⊗ ○, ⦶	no ○ below ⦷ and no ⦶ below ⊗

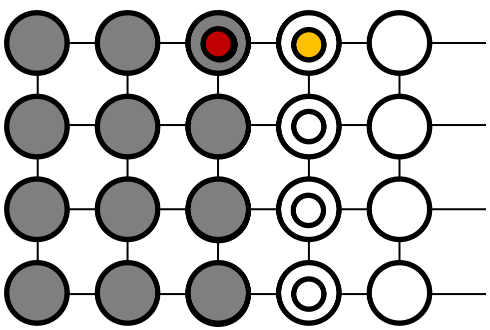
universality of adiabatic QC

2-local interactions, d=6 qudits

3 A local, sequential geometric clock in 2D

- 2D "sequential" evaluation

  [Aharonov van Dam Kempe Landau Lloyd Regev 04]



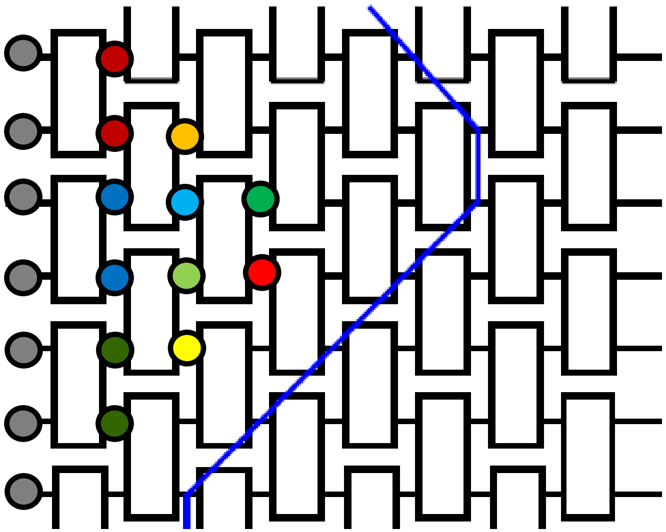
Forbidden	Guarantees that
○⦶, ○⦷, ○⊗	○ is to the right of all other qubits
○⊗, ⦶⊗, ⦷⊗	⊗ is to the left of all other qubits
○⊗, ⊗○	○ and ⊗ are not horizontally adjacent
⦶⦶, ⦶⦷, ⦷⦶, ⦷⦷	only one of ⦶, ⦷ per row
○/⦷, ⦶/⦷, ⊗/⦷	only ⦷ above ⦷
⦶/○, ⦶/⦷, ⦶/⊗	only ⦶ below ⦶
○/⊗, ⊗/○	○ and ⊗ are not vertically adjacent
⦷/○, ⊗/⦶	no ○ below ⦷ and no ⦶ below ⊗

universality of adiabatic QC

2-local interactions, d=6 qudits

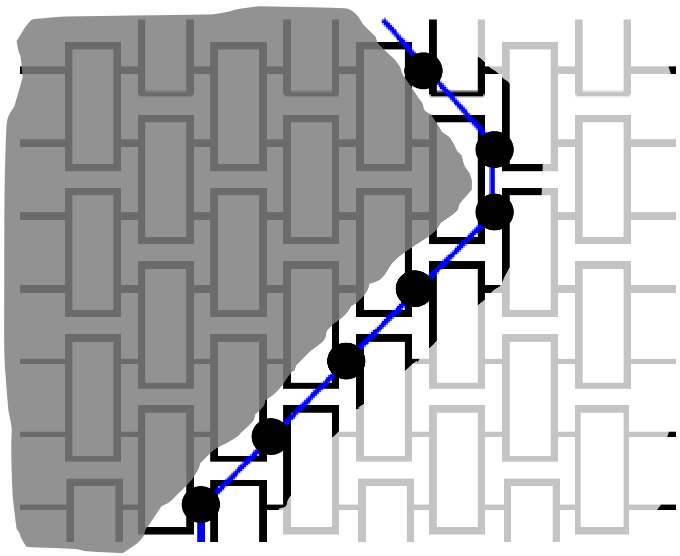
3 Another geometric clock in 2D: a string on a torus

- 2D "parallelized" evaluation [Mizel Lidar 06] [Janzing 07] [Breuckmann Terhal 13]



3 Another geometric clock in 2D: a string on a torus

- 2D "parallelized" evaluation [Mizel Lidar 06] [Janzing 07] [Breuckmann Terhal 13]



QMA-complete

4-local operations

b,b^{+} fermions, spin or d=4 (spin 3/2)

particle # tuning

(motivation: AQC)

- promise gap: proven $N^{-3}D^{-3}$, conjectured $N^{-2}D^{-2}=L^{-2}$

3 Constructing local, geometric clocks in 1D

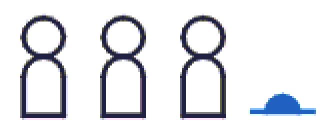
- moving the data with 2-local interactions

$$\left(|s\rangle - |t\rangle\right)\left(\langle s| - \langle t|\right)$$

$$|XY\rangle\langle XY|_{j,j+1} + |ZW\rangle\langle ZW|_{k,k+1}$$
$$- |PQ\rangle\langle NO|_{i,i+1} - |NO\rangle\langle PQ|_{i,i+1}$$

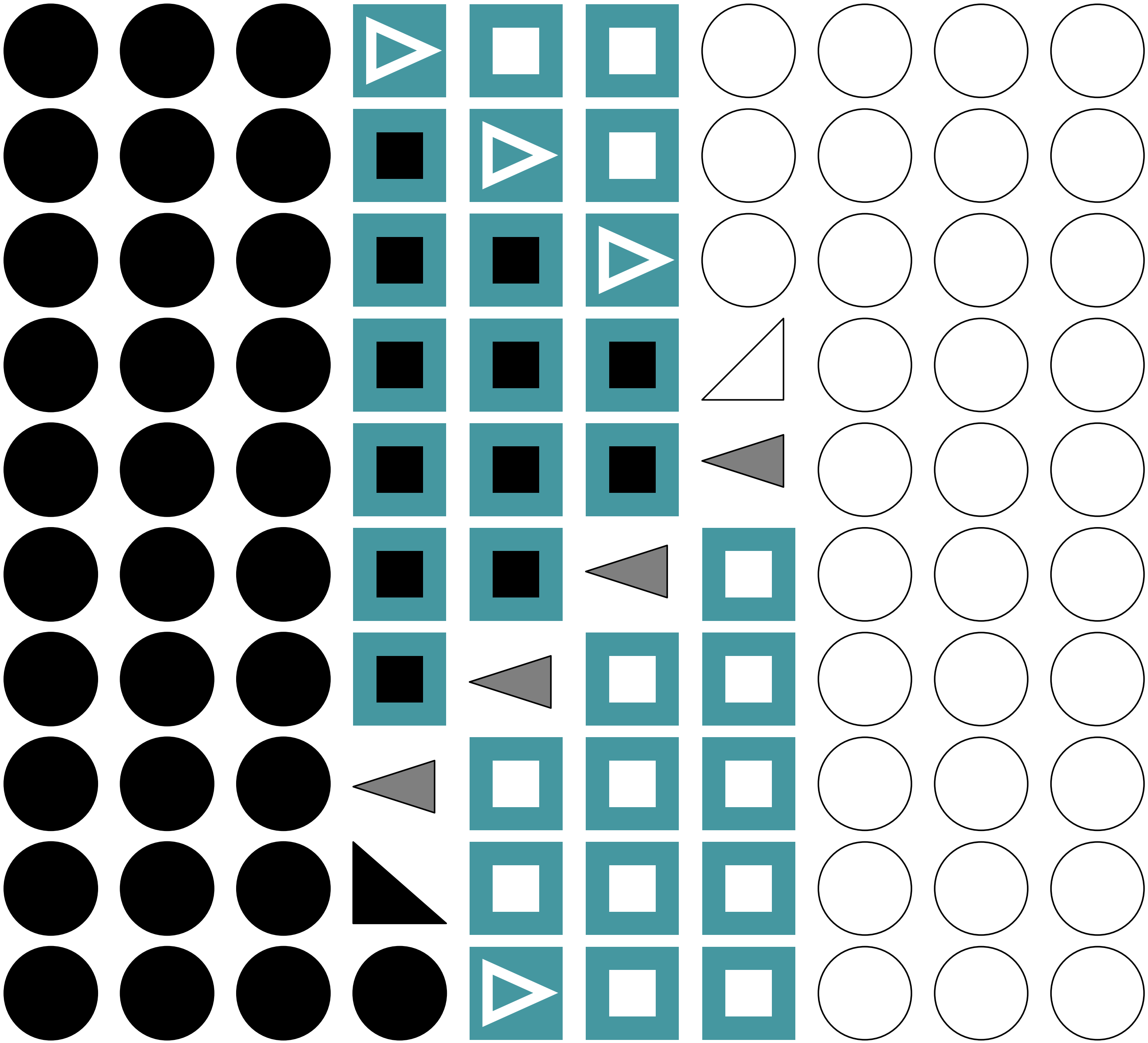
- higher local dimension: qudits

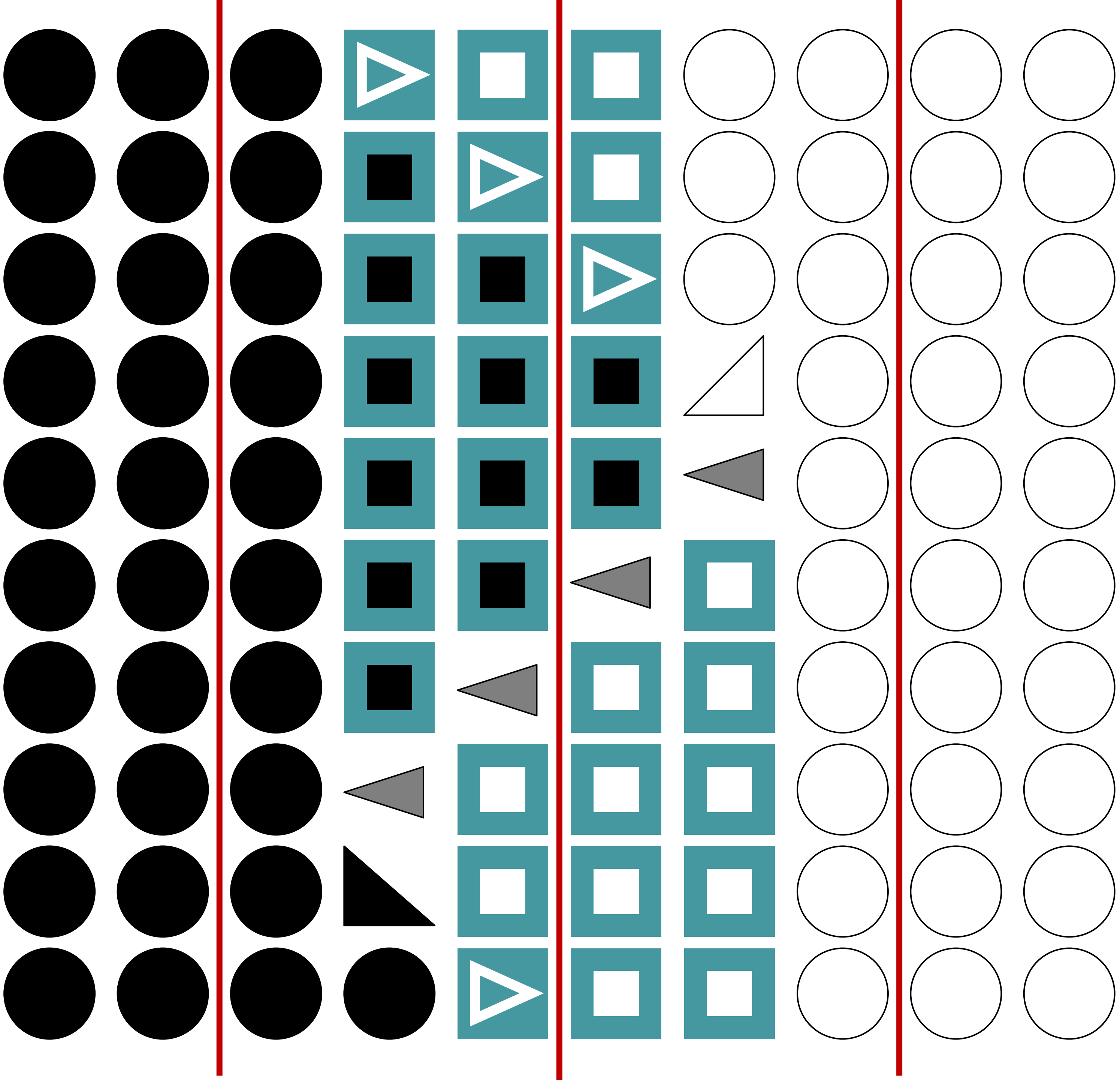
carry the data
mark transitions
detect bad states

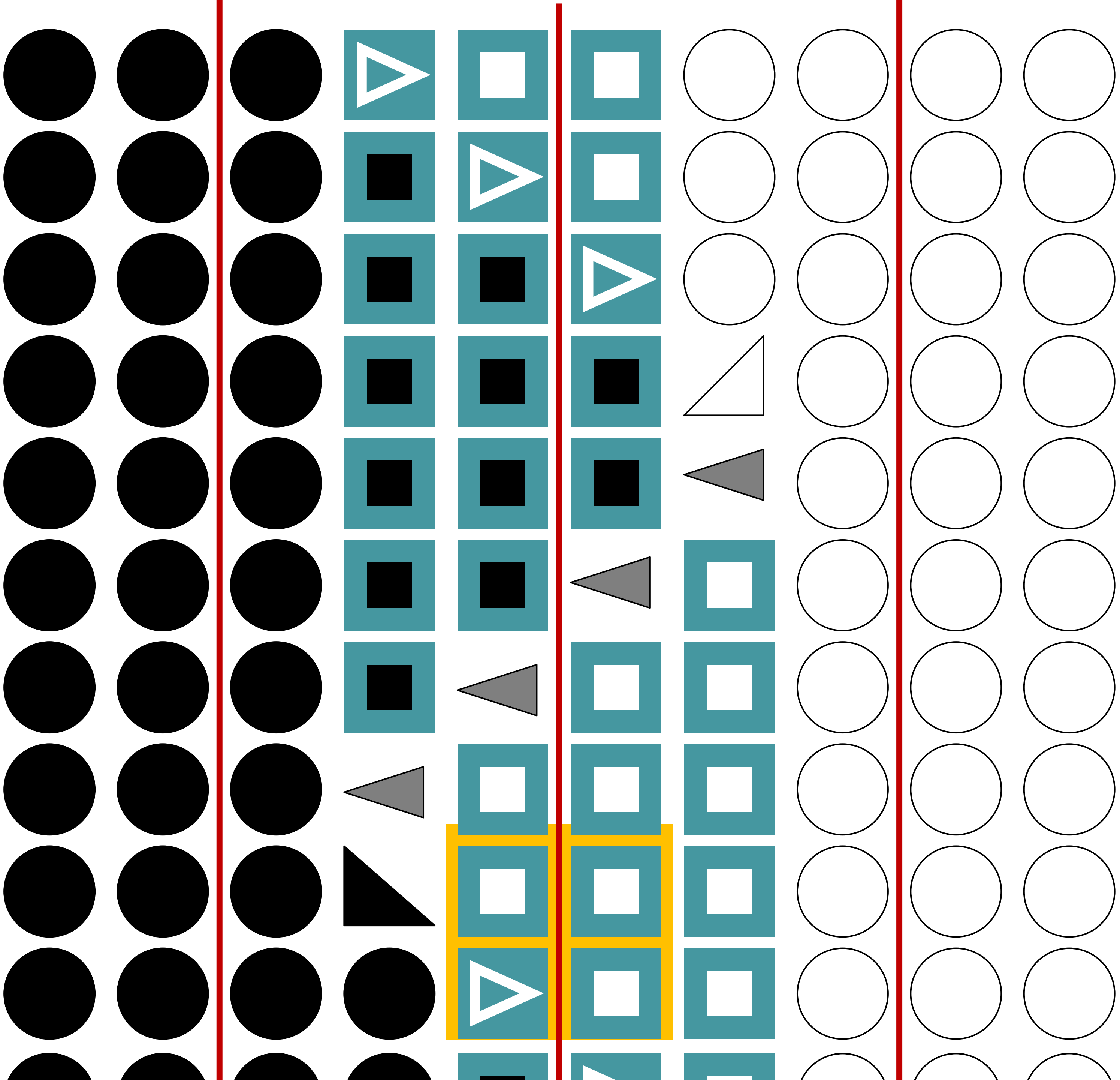


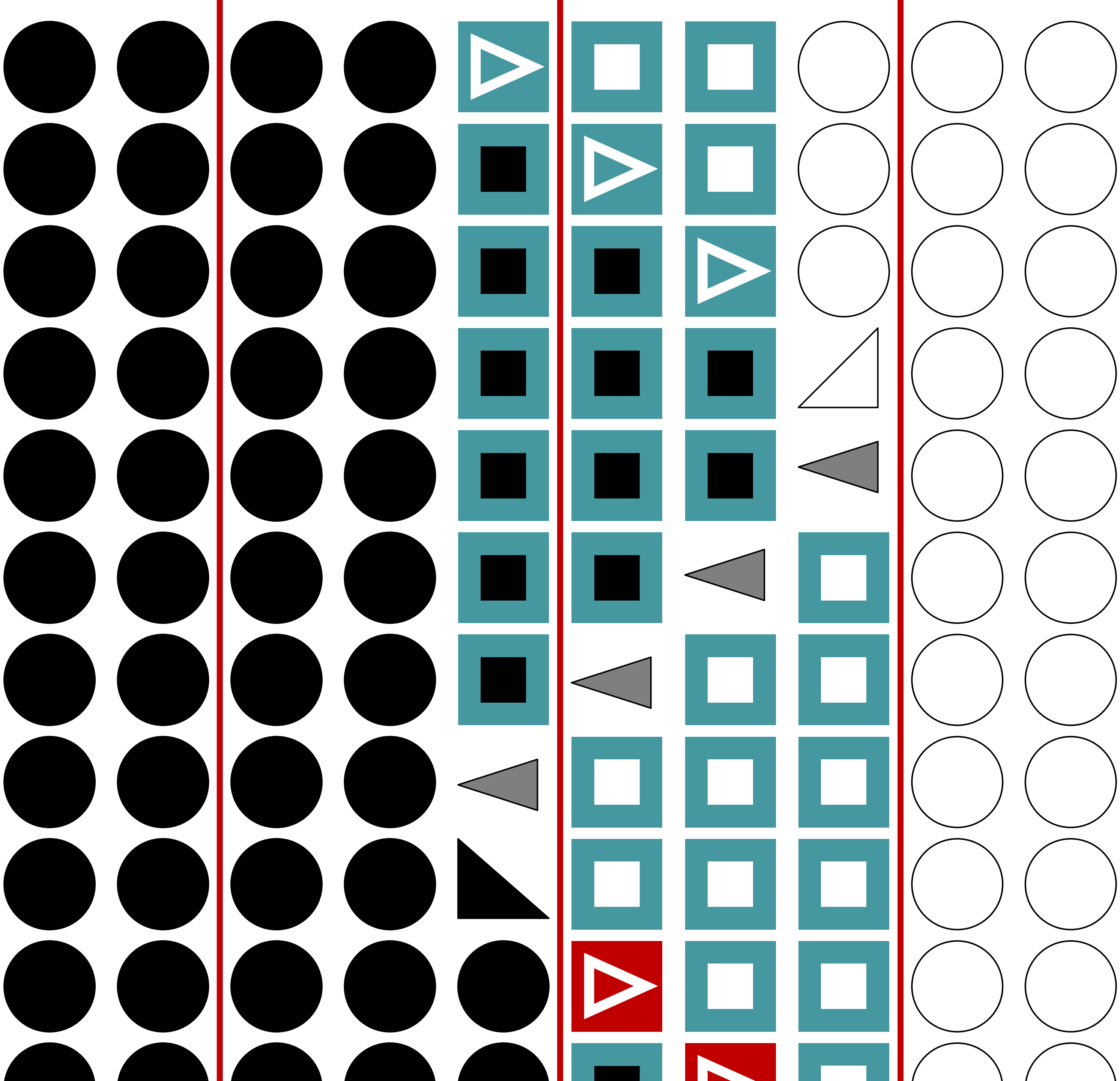
the **power** of quantum systems on a line

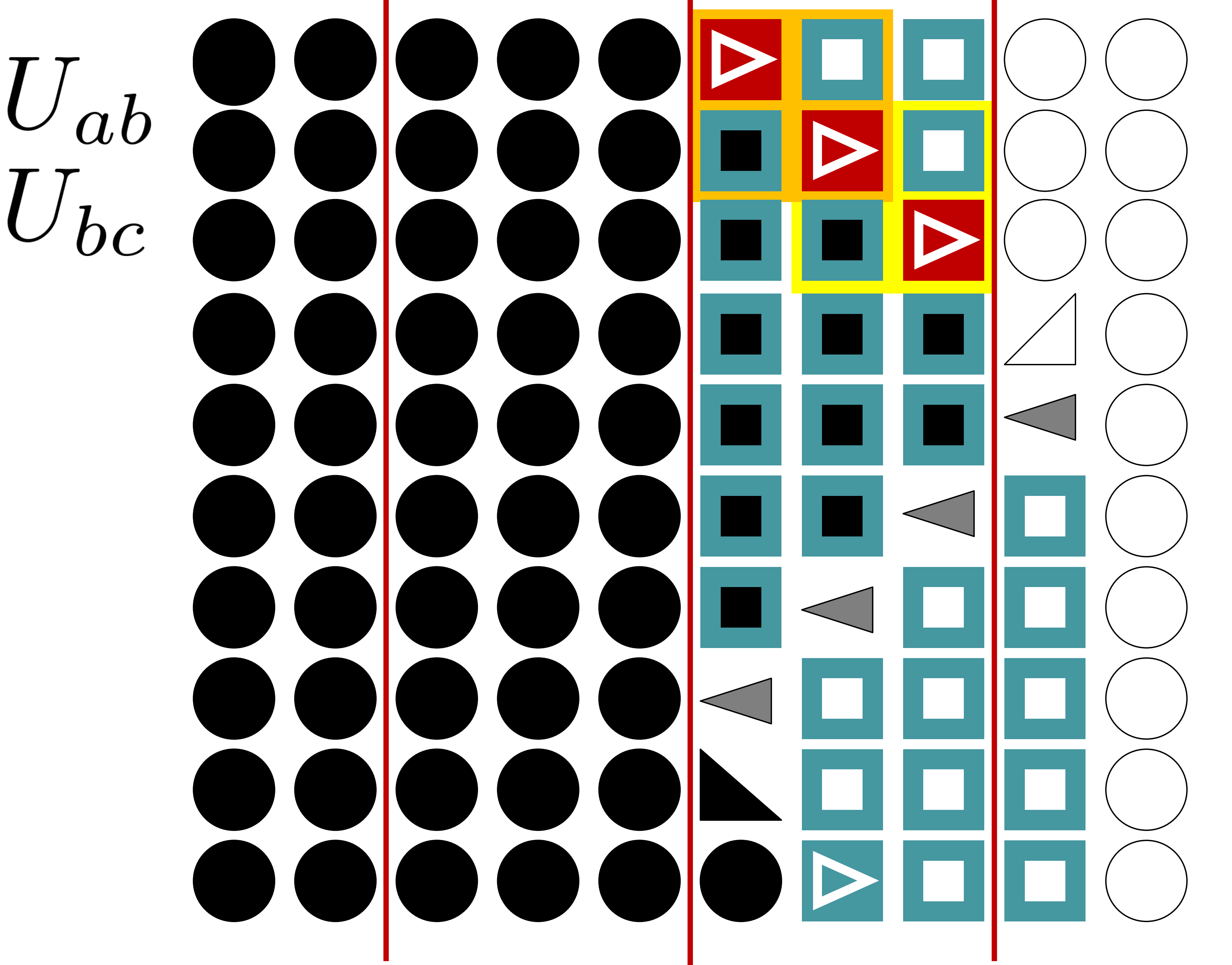
[Aharonov, Gottesman, Irani, Kempe]










U_{ab}
U_{bc}

1 LH in 1D (2-local) with qudits

- unique state progression

  every legal state goes
  to exactly 2 states

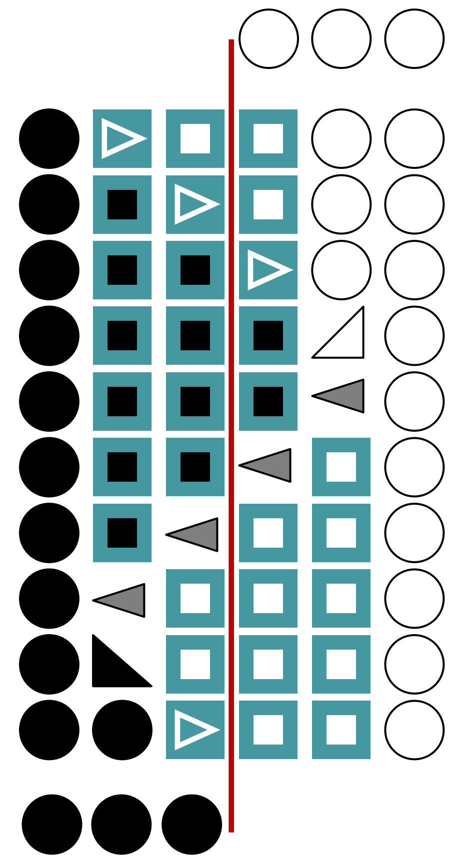
- clairvoyance

  allowed but illegal states
  evolve to forbidden ones

- the promise gap: L^{-3}

- an entangled ground state

  special case: NP-hard [Schuch]



QMA-complete

d=13

[AGIK '06]

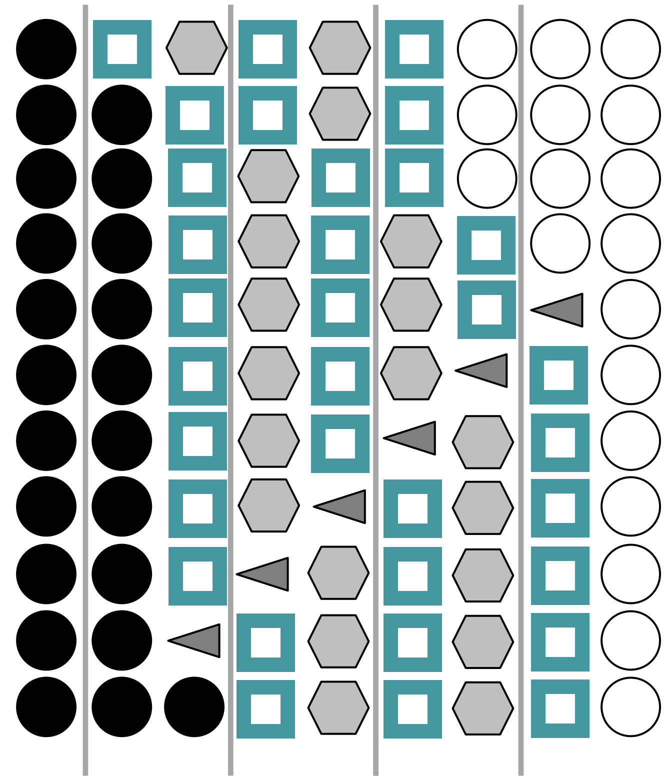
1 LH in 1D: more space = smaller qudits

- unique state progression

  every legal state goes to exactly 2 states

  d=11

  [N. 08]

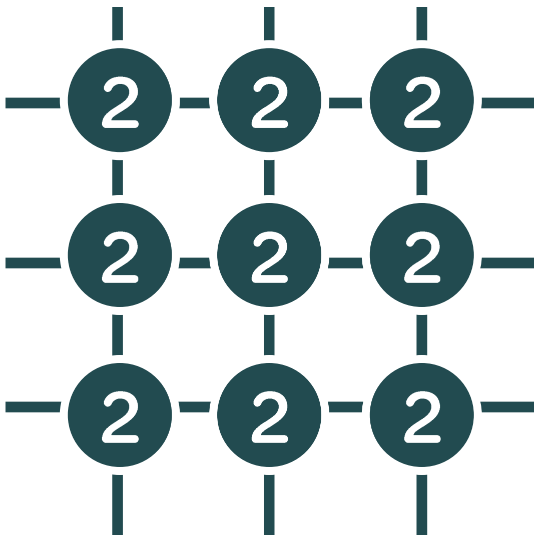


- bad but detectable transitions

  d=8

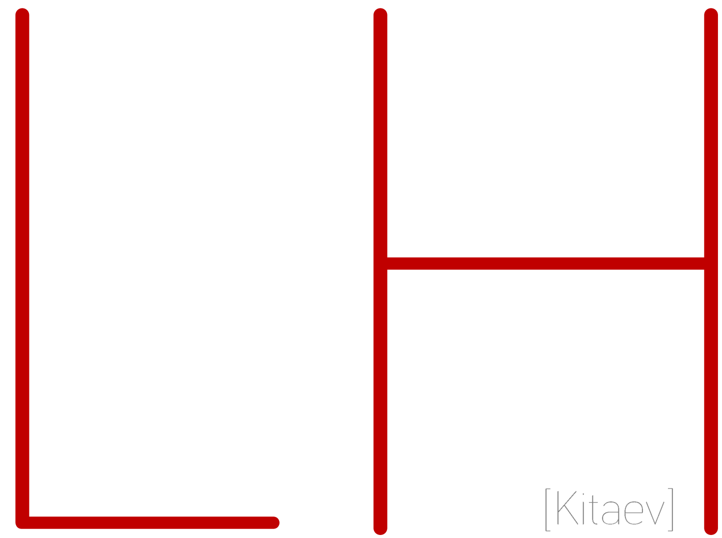
  [Hallgren, N., Narayanaswami '13]

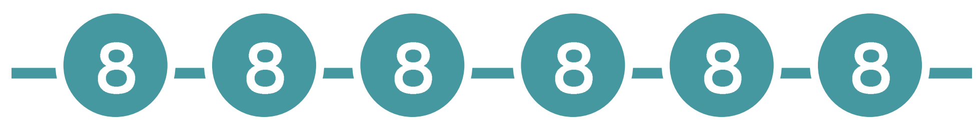
1 2-local Hamiltonian is QMA-complete



[Oliveira, Terhal '05]

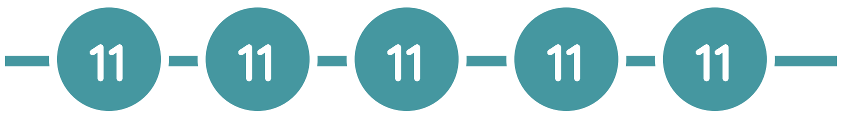
a global minimum



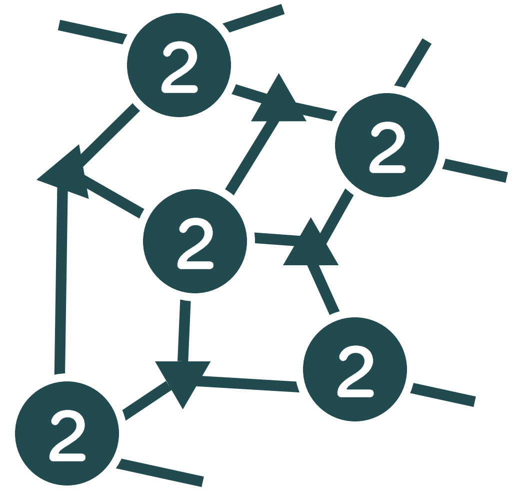


[Hallgren, N, Narayanaswami '13]

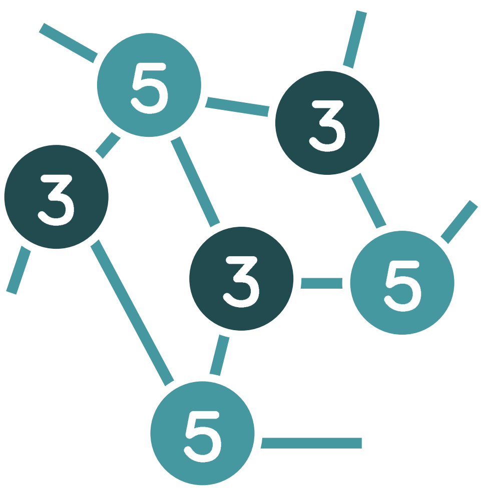
1 QMA$_1$-complete problems



[N. '08]



[Gosset, N. '13]



[Eldar, Regev '08]

unfrustrated

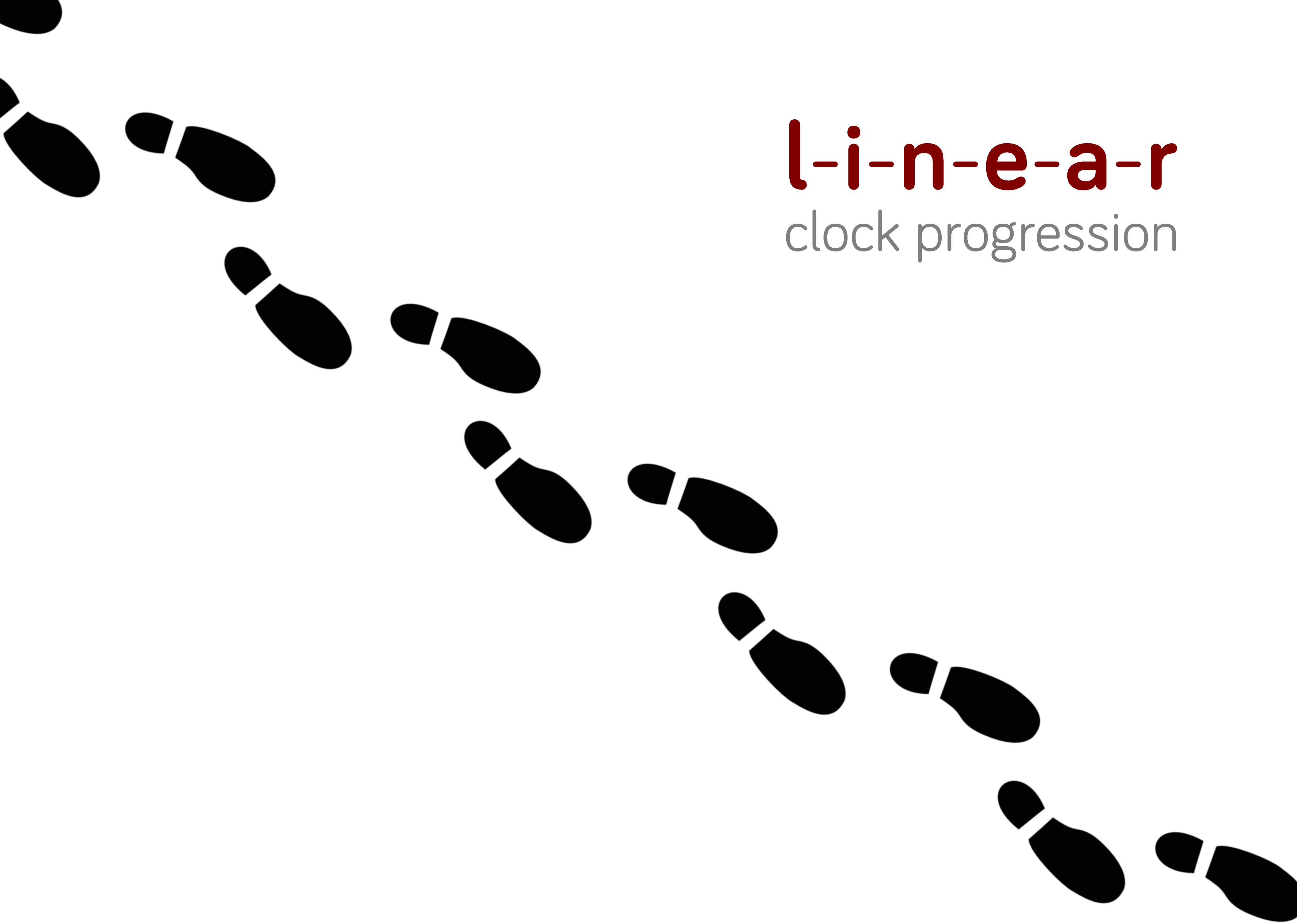
qSAT [Bravyi]

clock

constructing 2

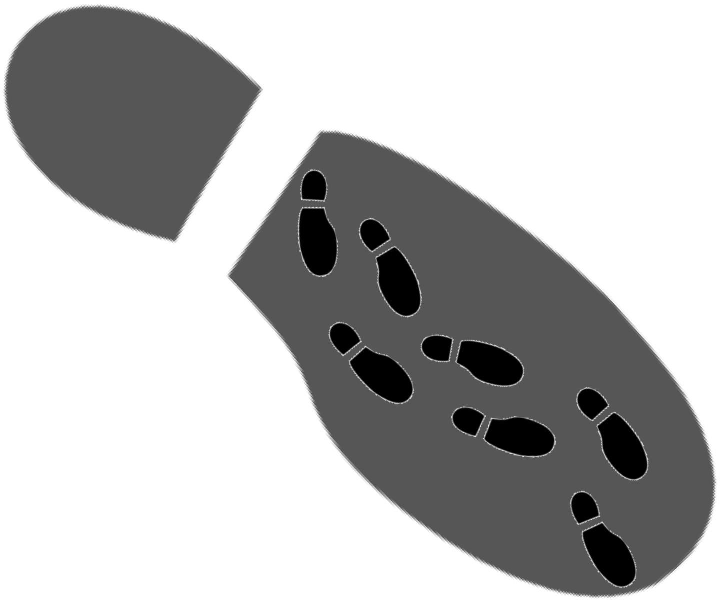
l-i-n-e-a-r

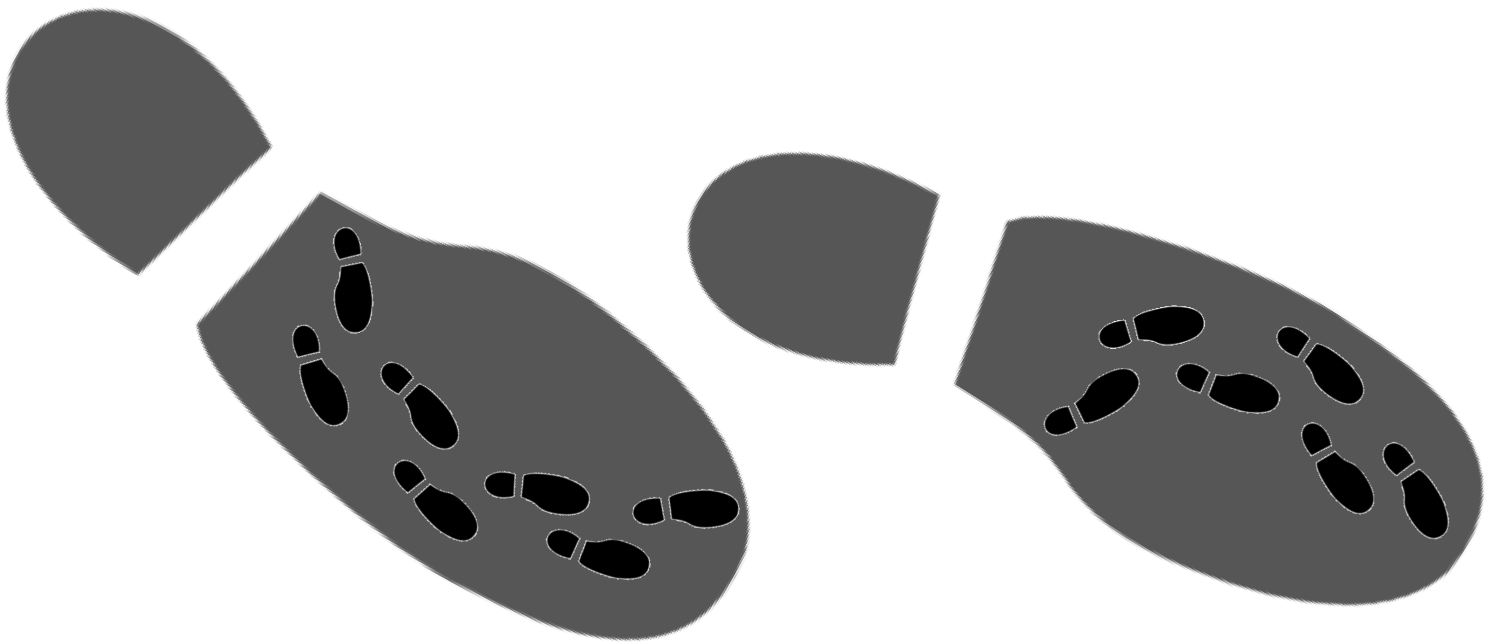
clock progression



com-pos-ite

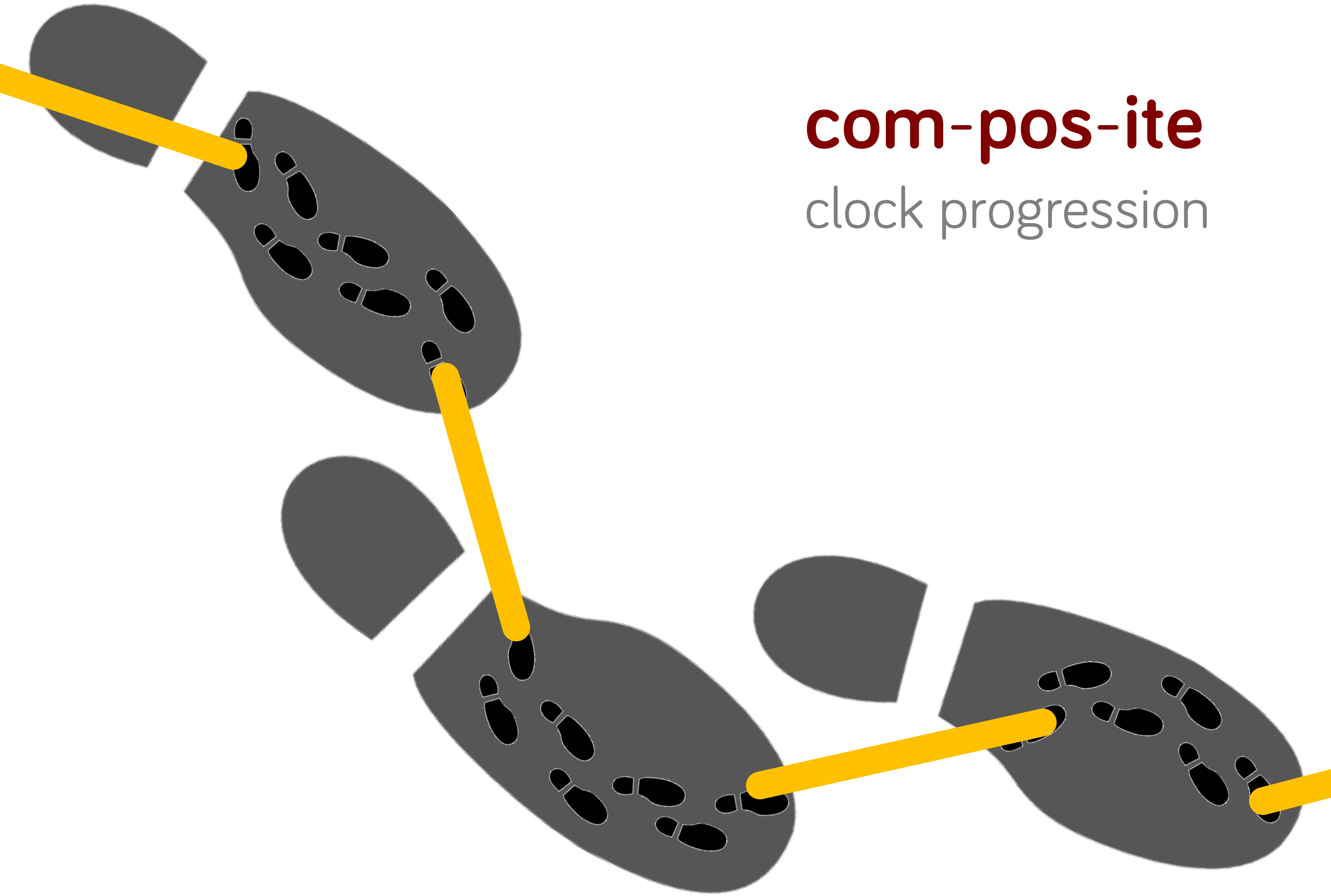
clock progression





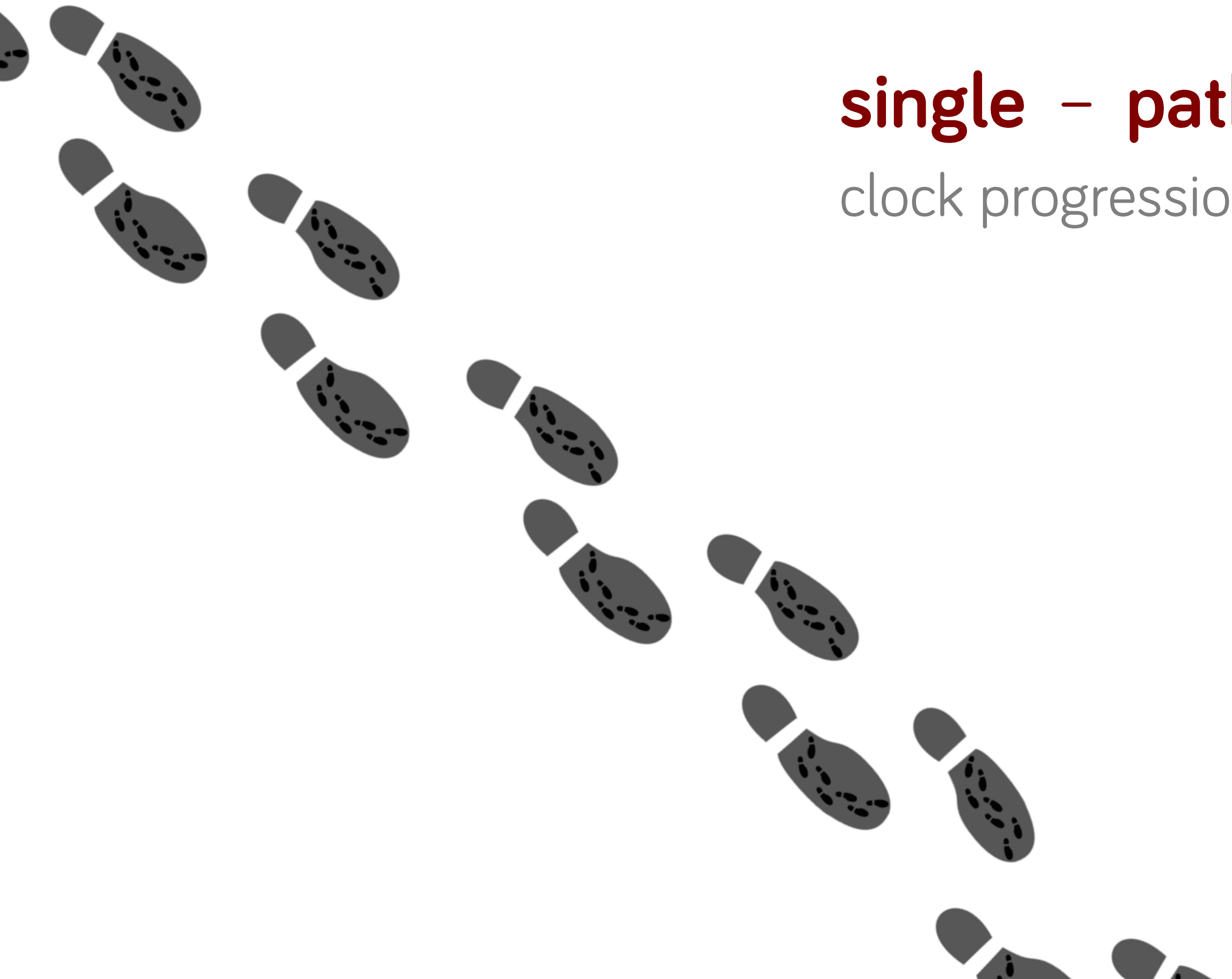
com-pos-ite

clock progression



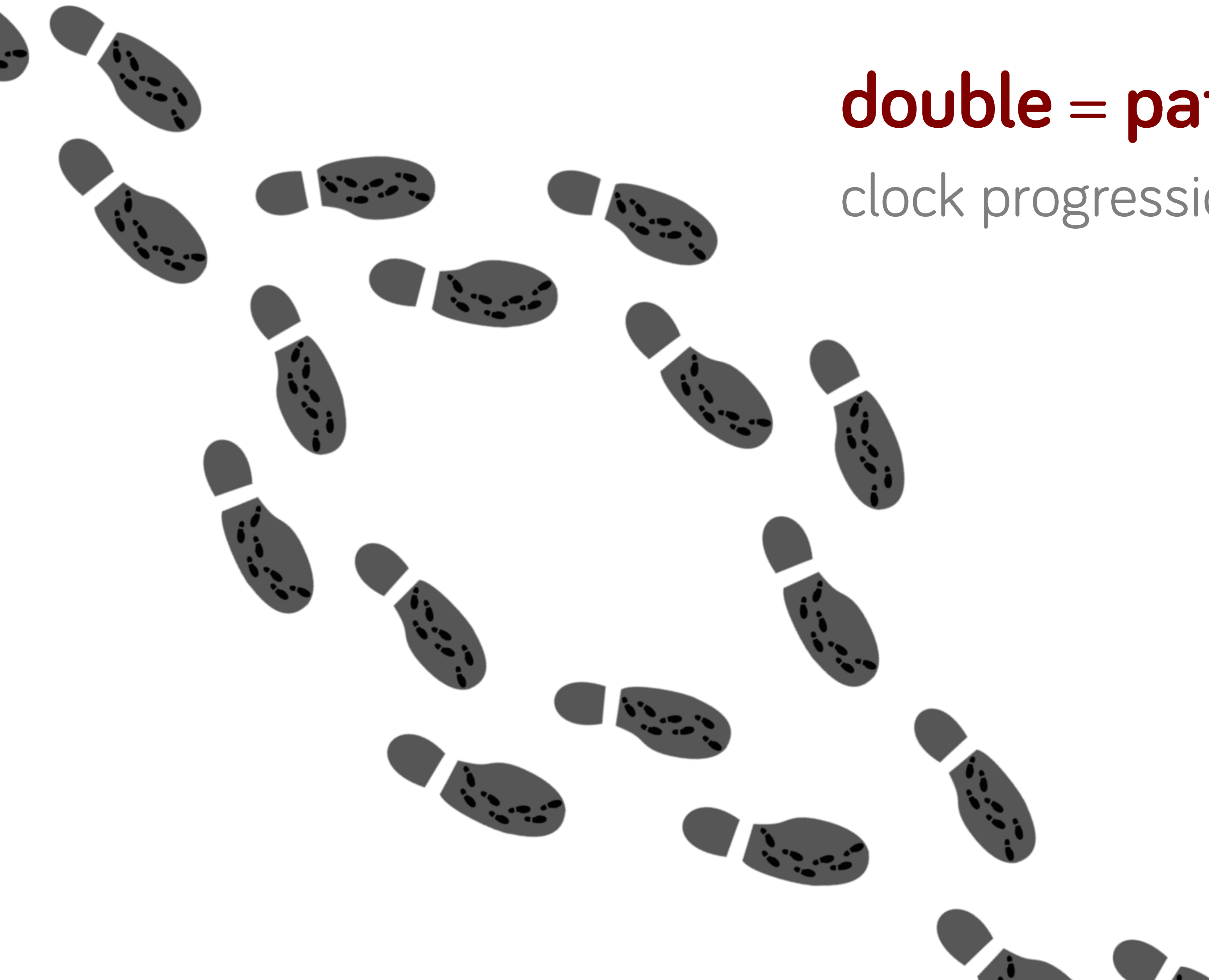
single – path

clock progression



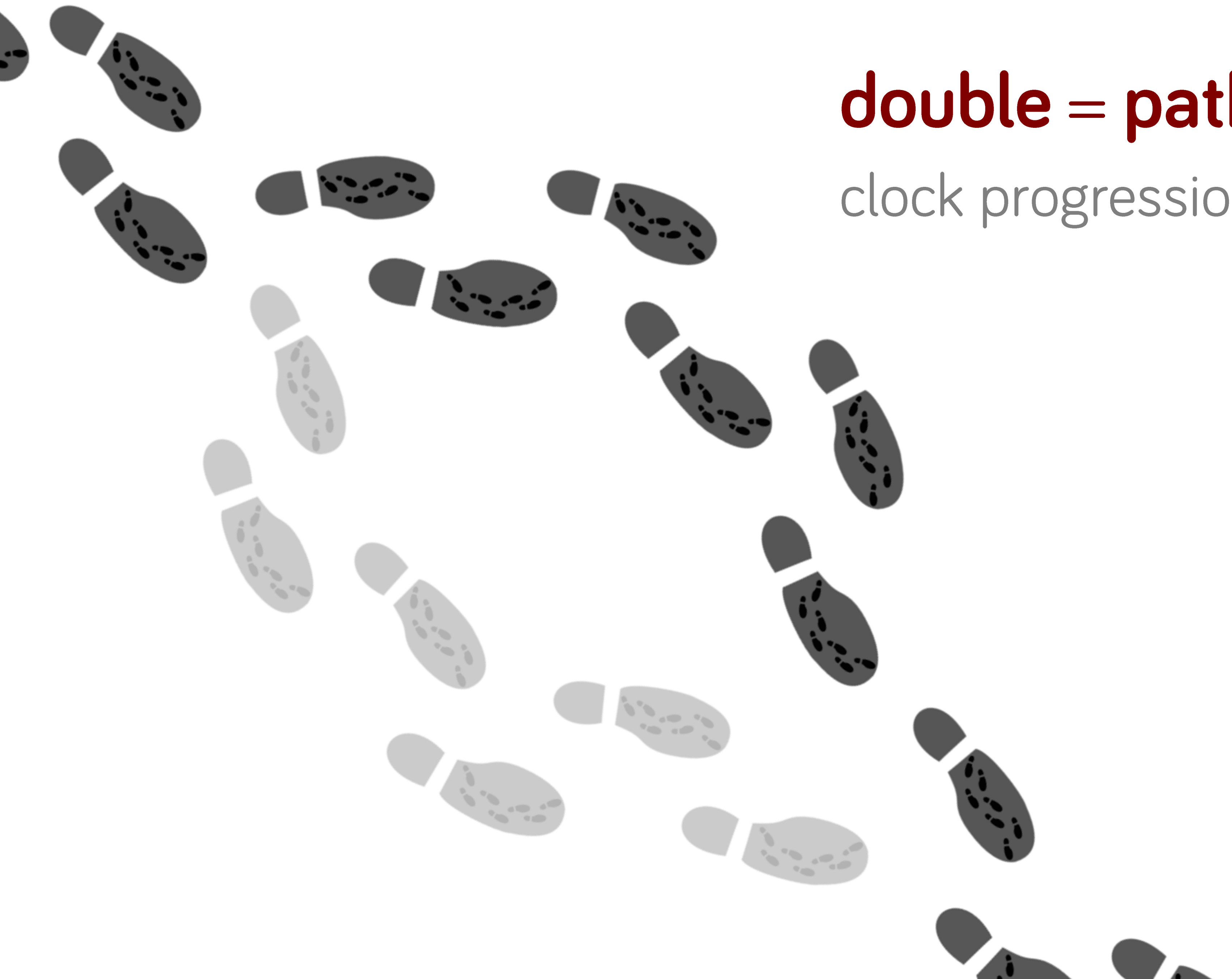
double = path

clock progression



double = path

clock progression

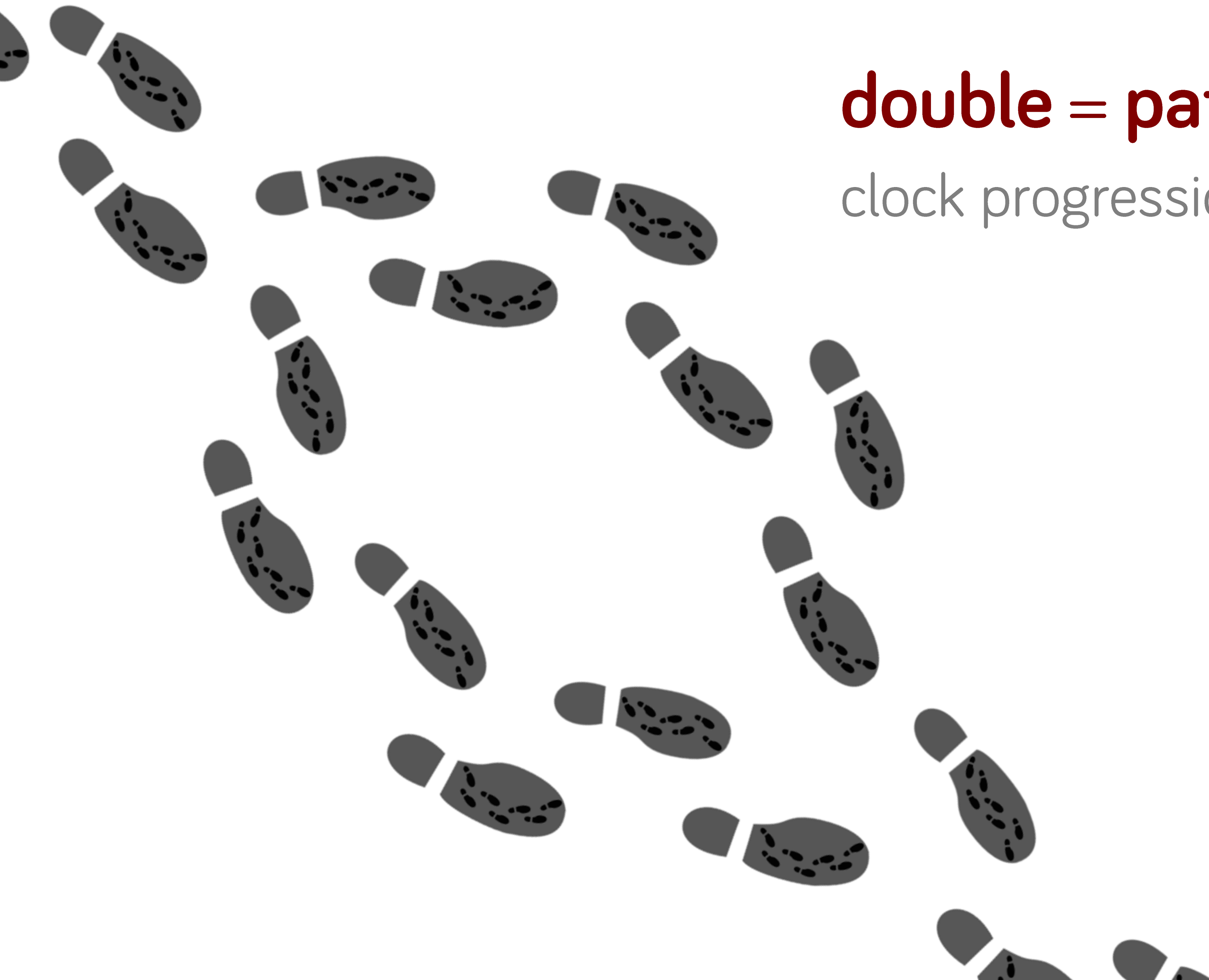


double = path

clock progression

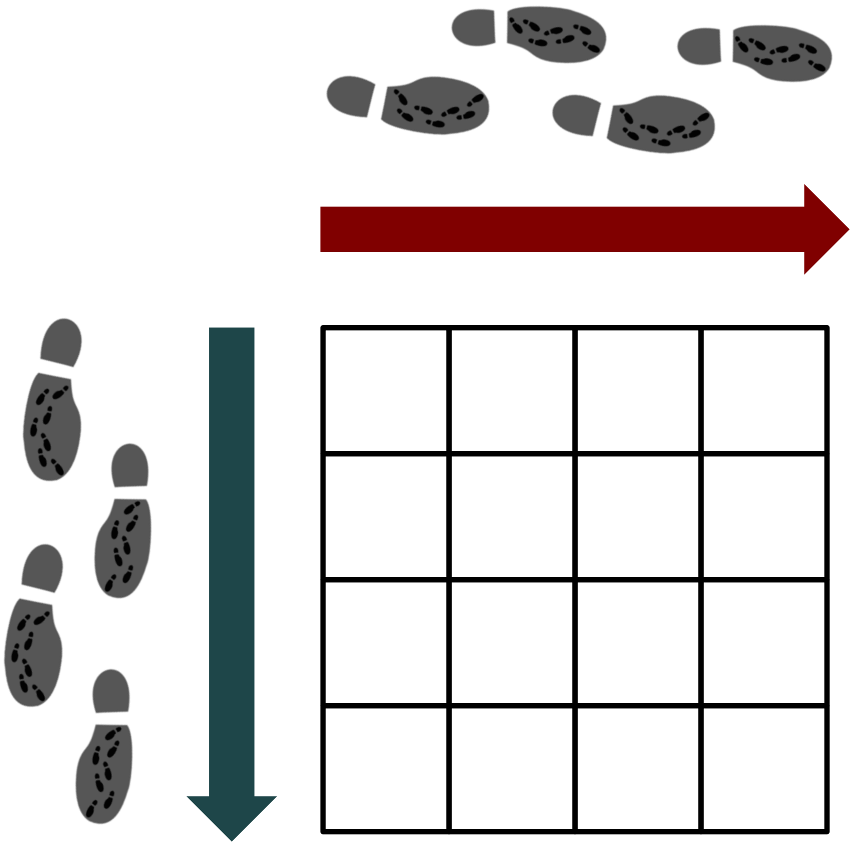
double = path

clock progression



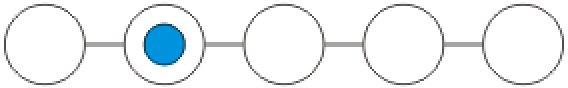
2 clocks: 2D

clock progression



5 Applying 2-qubit gates 3-locally

- the railroad switch



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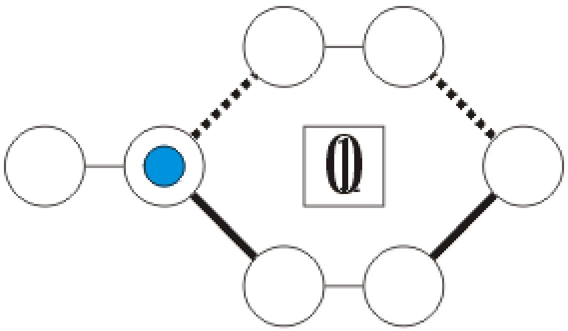
- the railroad switch



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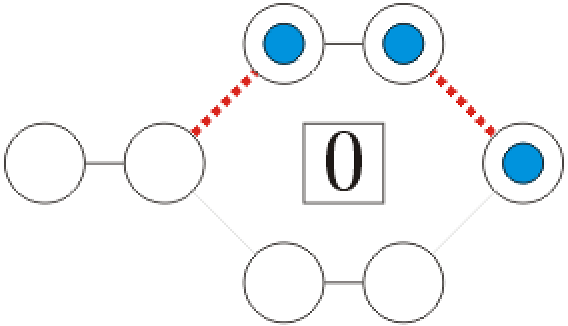

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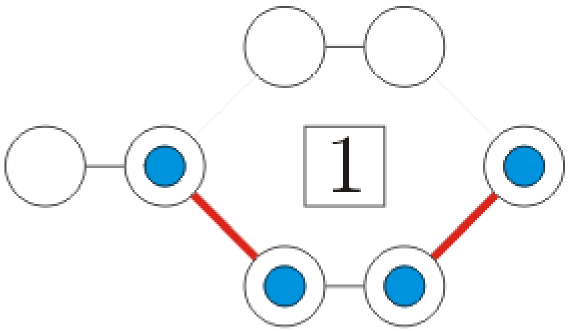

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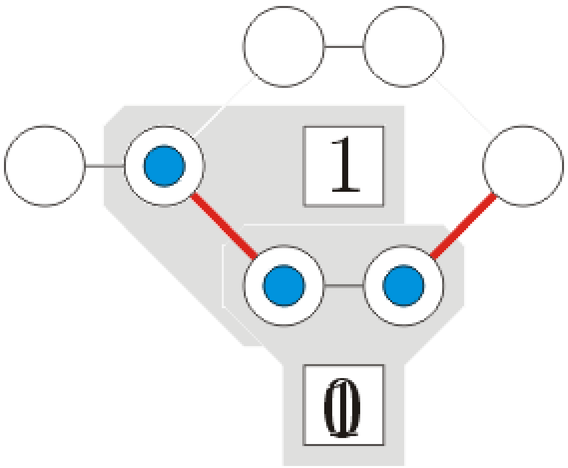
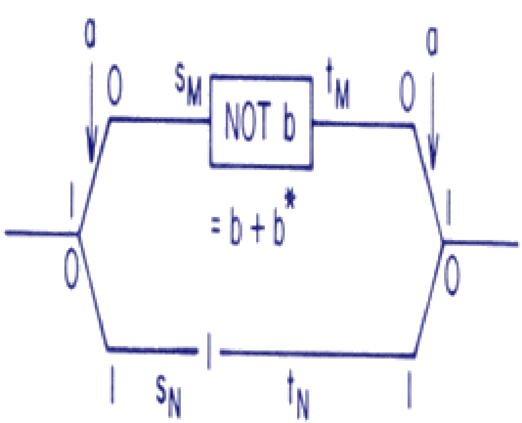


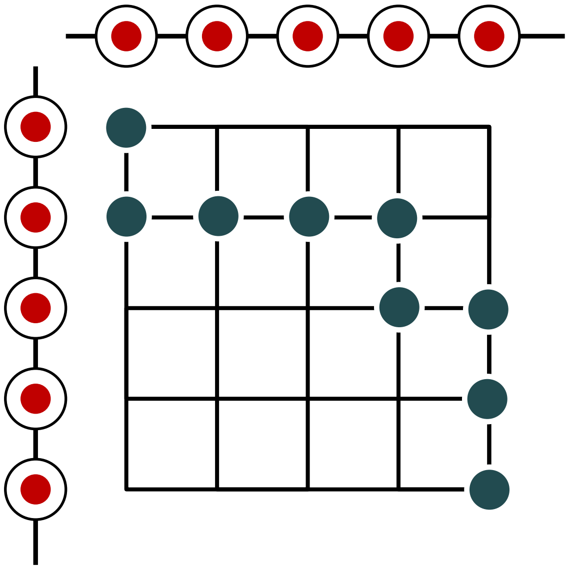
CNOT: 3-local

needs initialization



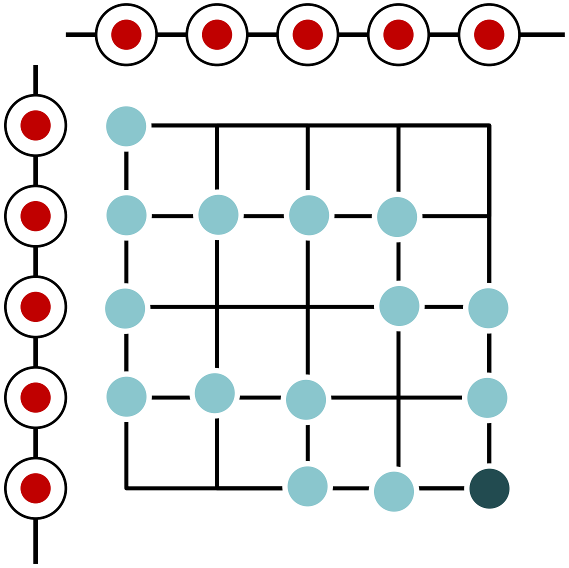
5 2D clocks (with two registers)

- two clocks



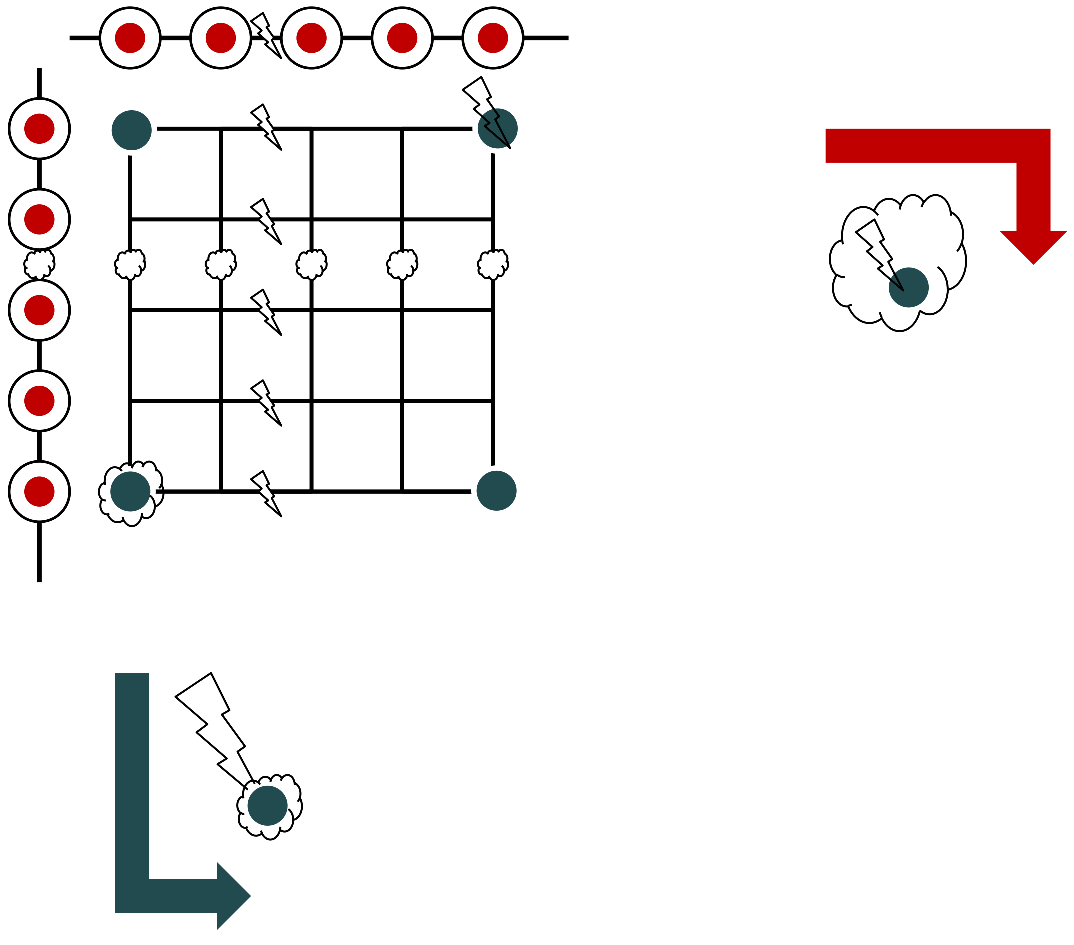
5 2D clocks (with two registers)

- two clocks



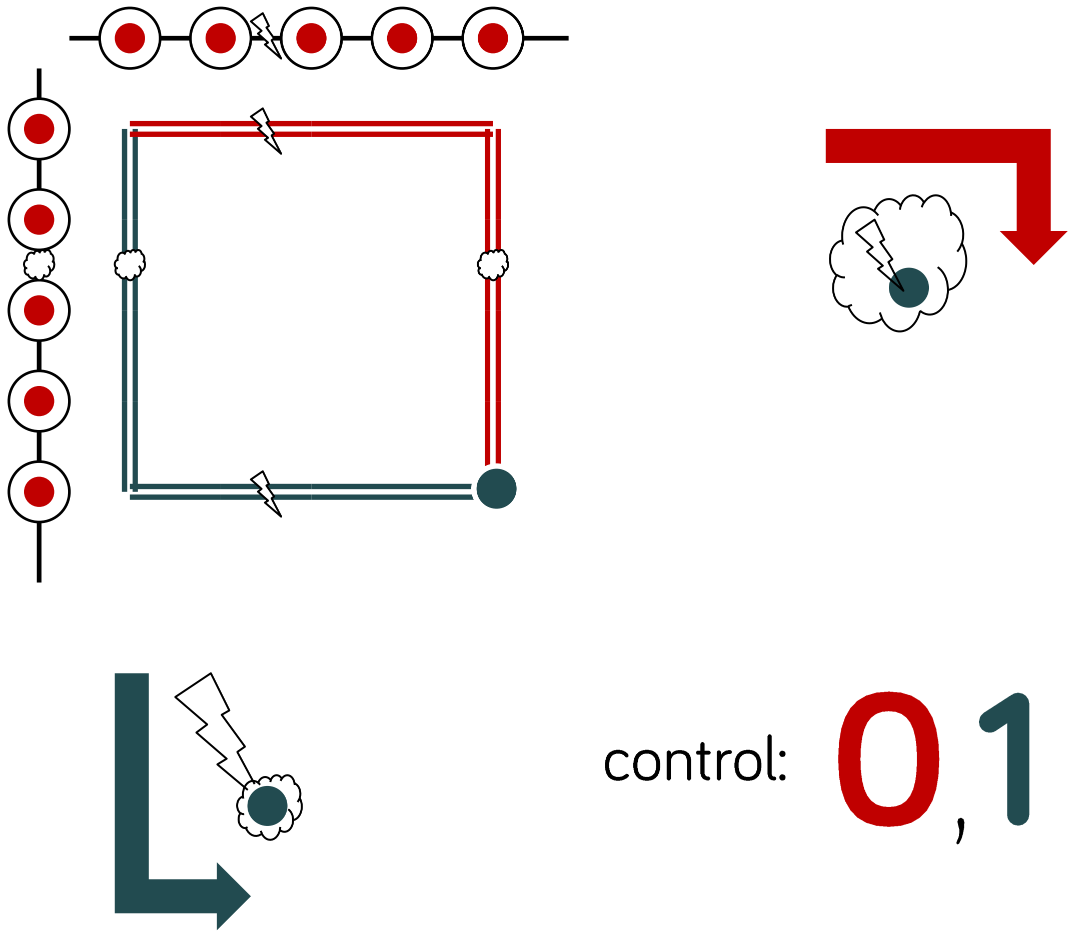
5 2D clocks (with two registers)

- add non-commuting (data) operations



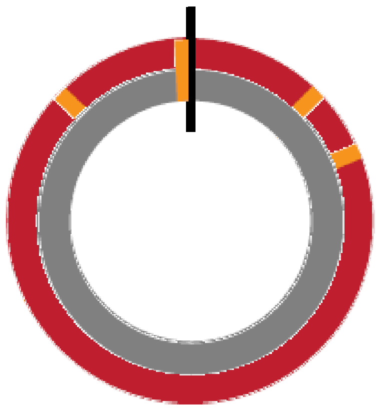
5 2D clocks (with two registers)

- like a railroad switch… with a single active site ensured

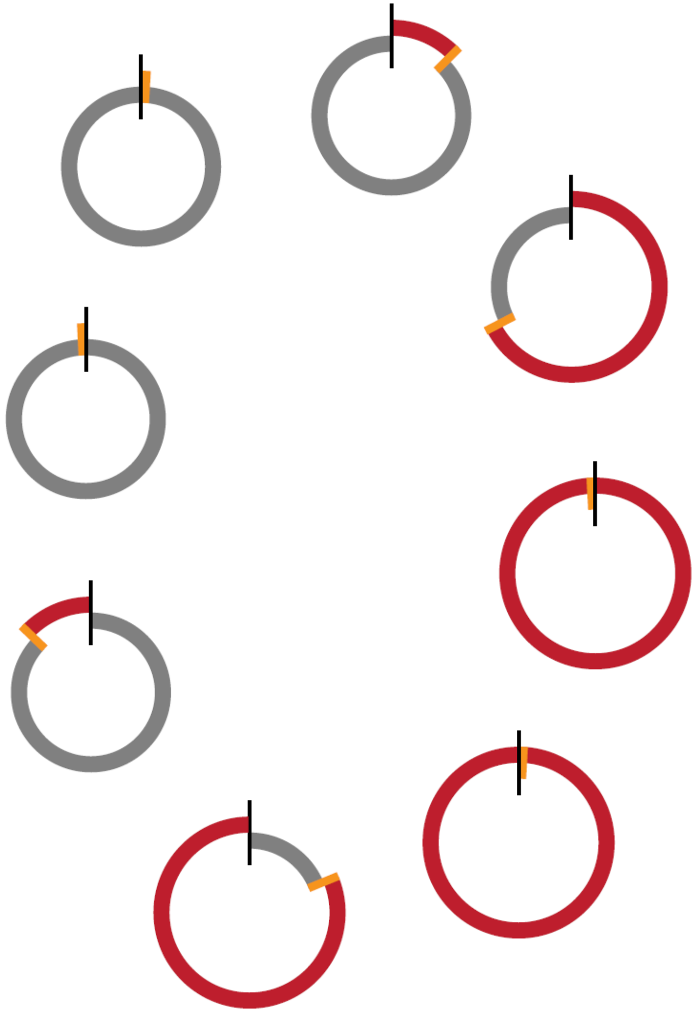


q3-SAT
QMA$_1$-comp.
[Gosset, N. '13]

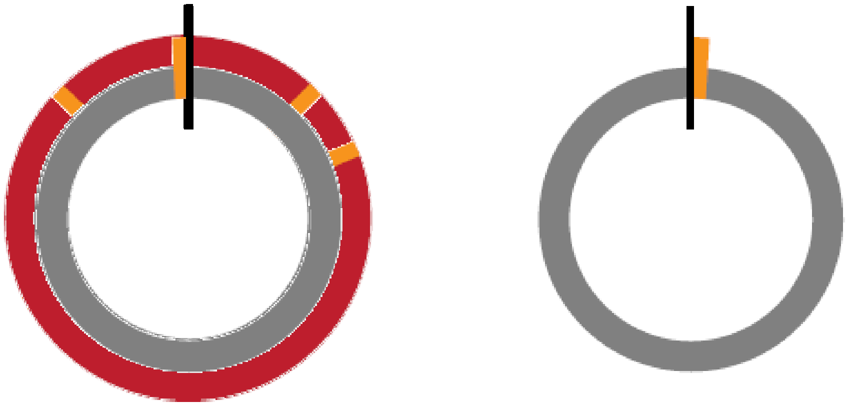
3 Can a clock be shorter than unary?



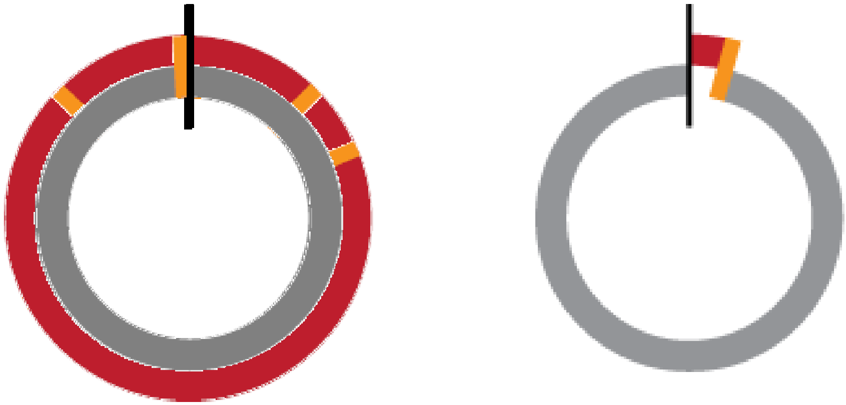
- a qutrit surfer on a cycle: 2C states



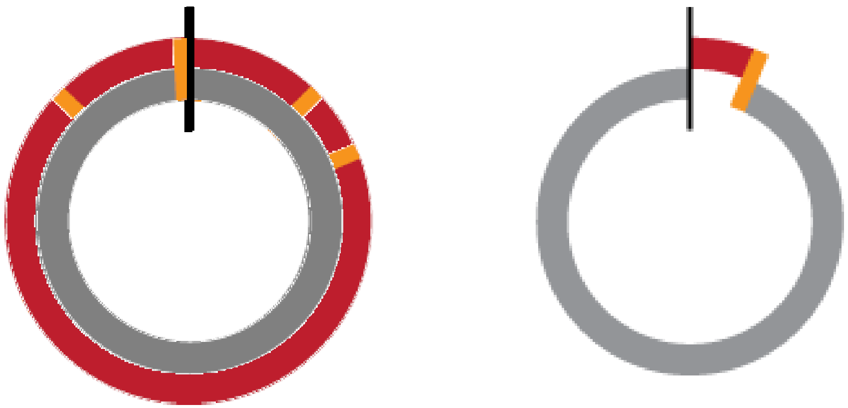
3 A smaller clock using two coupled cogs



3 A smaller clock using two coupled cogs



3 A smaller clock using two coupled cogs



- 2 cogs of length C give us $(2C)^2$ clock states
- transitions: 4-local, gates: 6-local (can be improved)
- the promise gap for a circuit with L gates: still L^{-2}

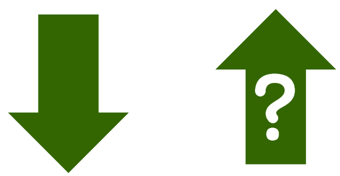
qPCP
& clocks

3 Questions about the qPCP conjecture

[Martin Schwarz]

- equivalence of the two formulations? [AAV13]

LH with a fractional promise gap



translating the *random* small verification to a LH?

look at a few qubits of a proof

- locally checking the (expected) very entangled states? [D.Aharonov, L.Eldar]

3 Questions about the qPCP conjecture

- clock constructions have a 1/poly promise gap

  consistent, effective interaction strengthening?
  error-correction/detection based quantum gadgets?

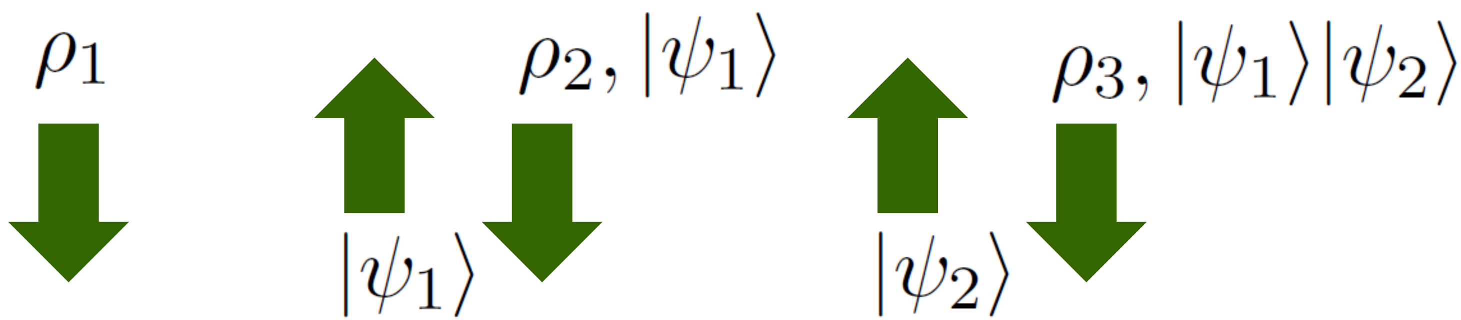
- direct Hamiltonian methods?

  beyond the history state? [Itai Arad]

$$M := \mathbb{I} - H/m$$

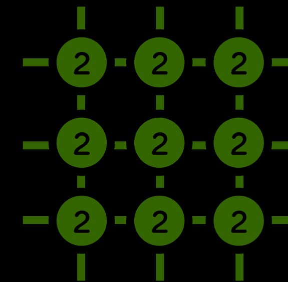
$$\mathrm{Tr}(M^\ell) \qquad \begin{aligned} \Gamma &= 1/\mathrm{poly}(n) \\ \ell &= \Omega(mn/\Gamma) \end{aligned}$$

an interactive protocol to check the trace:



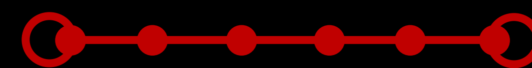
1 **strong promises**

and eigenvalue gaps



2 **the history**

of the history state



3 **running the clock**

precise/faulty, qubit/qudit, sequential/parallel

4 **on the qPCP road**

questions & warnings