Approximation power of deep networks

Matus Telgarsky <mjt@illinois.edu>

(with help from many friends!)

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replace $f : \mathbb{R}^d \to \mathbb{R}$ with neural network $g : \mathbb{R}^d \to \mathbb{R}$.

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$$\int \ell(f(x), y)\,\mathrm{d}P(x, y) \qquad \text{vs.} \qquad \int \ell(g(x), y)\,\mathrm{d}P(x, y).$$

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- **Upper bounds:** If $\ell(\cdot, y)$ is 1-Lipschitz,

$$\int \left[\ell(g(x), y) - \ell(f(x), y)\right] \mathrm{d}P(x, y) \leq \left|g(x) - f(x)\right| \mathrm{d}P(x, y);$$

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  we make this small everywhere **(universal/uniform/$L_\infty(P)$ apx)**, or in $L_1(P)$.
- **Lower bounds:** we want large error on a large set; as a surrogate, $|g - f|$ large in $L_1(P)$ or $L_1(\mathrm{Unif})$.

By **deep networks** we mostly mean

$$x \mapsto A_L \sigma_{L-1}\left(\cdots \sigma_1(A_1 x + b_1) \cdots\right) + b_L,$$

where nonlinearity/activation/transfer σ_i
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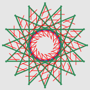
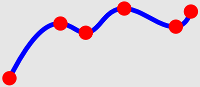
We'll mostly stick to the ReLU $z \mapsto \max\{0, z\}$ (Fukushima '80);
it's easy to convert.



Elementary universal approximation.



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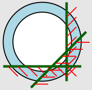
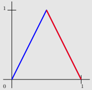


Benefits of depth.



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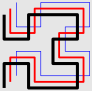
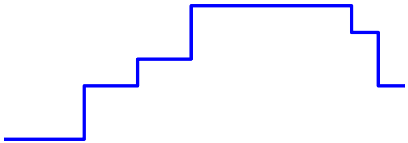


Odds & ends.

Univariate functions via step activations



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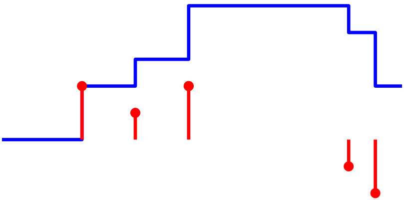

$$x \mapsto 2\cdot\mathbb{1}[x-3 \geq 0] + \mathbb{1}[x-5 \geq 0] + 2\cdot\mathbb{1}[x-7 \geq 0] - \mathbb{1}[x-13 \geq 0] \cdots$$

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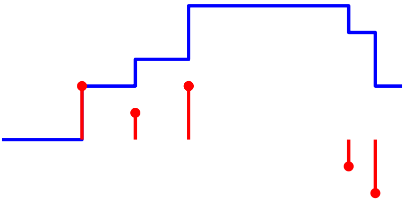

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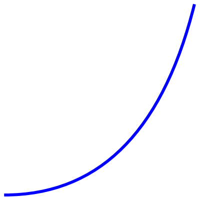
Remark. By contrast, polynomials struggle with flat regions.

Smooth univariate functions via step activations



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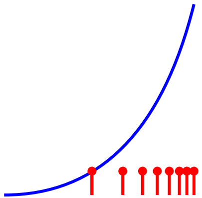

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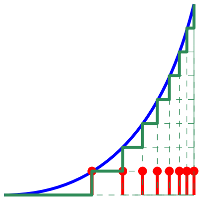

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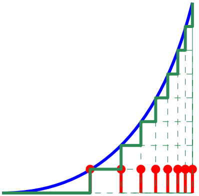

Approach #1: subdivide range, $^{\text{Lip}}/_{\epsilon}$ steps.

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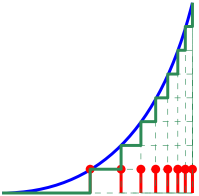

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Approach #2: by FTC, for $x \geq 0$,

$$f(x) = f(0) + \int_0^x f'(b)\,\mathrm{d}b = f(0) + \int_0^\infty \mathbb{1}[x - b \geq 0] f'(b)\,\mathrm{d}b.$$

This is a density over infinitely many steps/nodes!
Sample $^{\text{avg Lip}}/_{\epsilon^2}$ steps.

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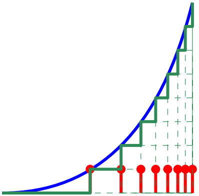

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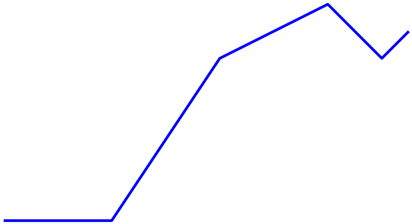
Remarks.

- Infinite width network!
- Refined average-case estimate! (Captures flat regions.)

Univariate functions via ReLU activations



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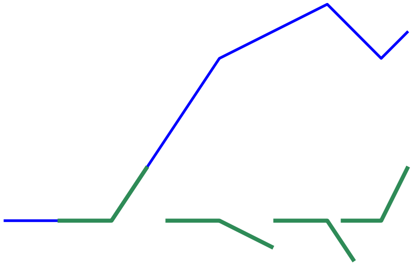

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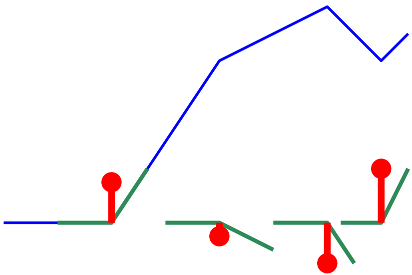

Include ReLU $z \mapsto \max\{0, z\}$ with change of slope.

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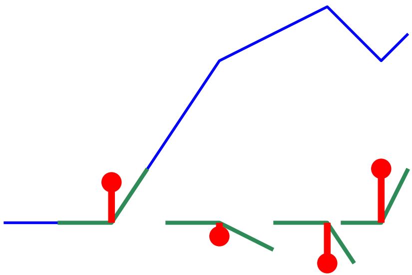

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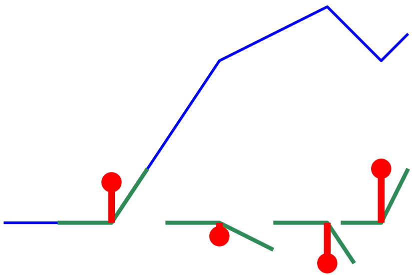

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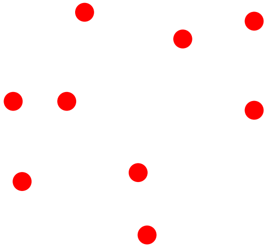
(In some sense optimal (Savarese-Evron-Soudry-Srebro '19).)

Multivariate, but finitely many points



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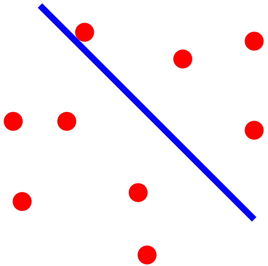

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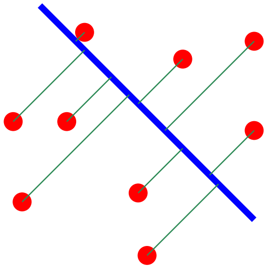

With probability 1, a random line has unique projections.
We've reduced to the univariate case.

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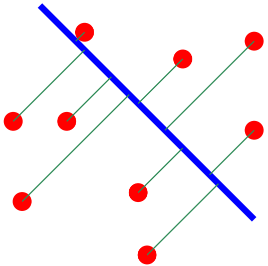

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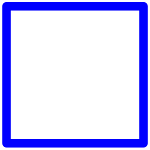
Caveats:

- Representation size may have blown up.
- Not our original goal.

Approximate a multivariate box.



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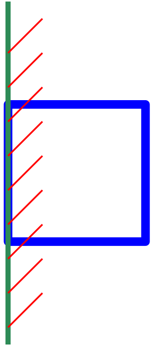

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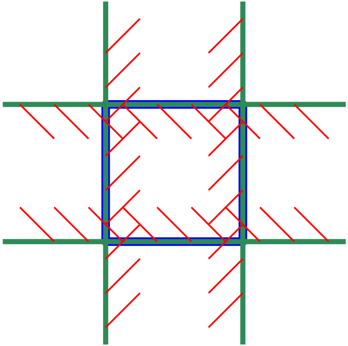

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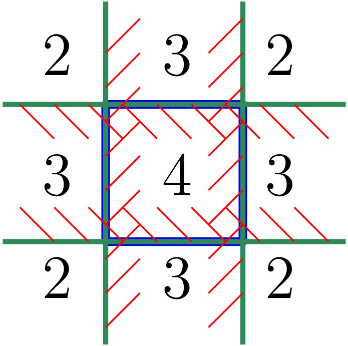

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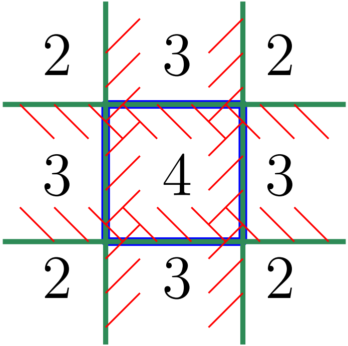

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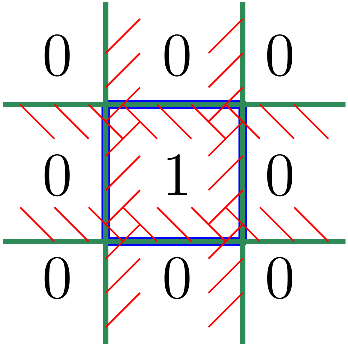

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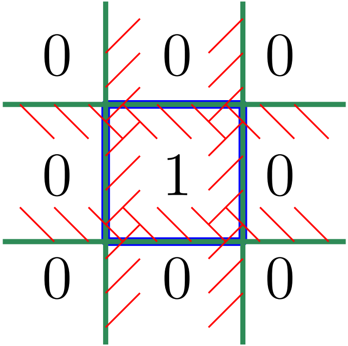

Supporting hyperplanes! **...oops.**

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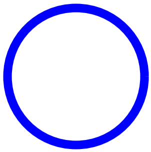
Fix #2: add a layer, thresholding at 3.5!
...how about one ReLU/hidden layer?

Approximate a multivariate ball.



Fix #3: add all the hyperplanes!

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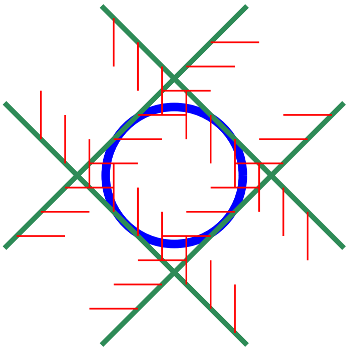

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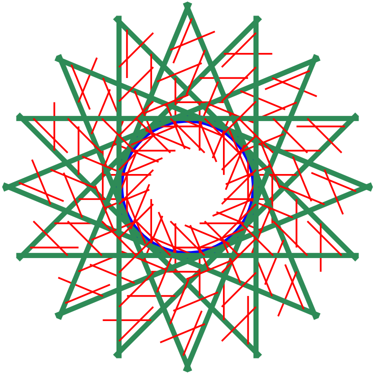

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Resulting radial function is constant within ball, attenuates away from it.

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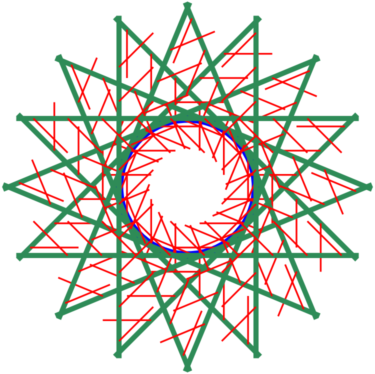

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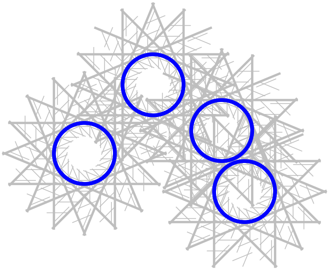
Bad news: good apx seems to require 2^d nodes…
(We'll come back to this.)

Combinations of radial bumps.



Normalize bumps/RBFs into density p; convolve with f.

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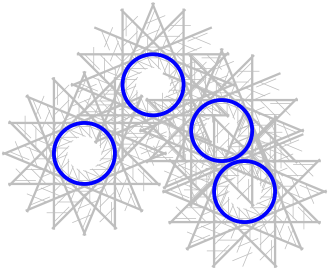

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$$\left|f(x) - \int f(z)p(x-z)\,\mathrm{d}z\right| = \left|f(x) - \int f(x-z)p(z)\,\mathrm{d}z\right|$$

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which is small if $p(z) \approx 0$ for large $\|z\|$.

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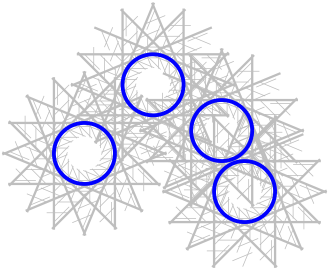

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Size estimate: $\left(d\cdot\mathrm{Lip}/\epsilon\right)^{\mathcal{O}(d)}$.

(Mhaskar-Michelli '92, BJTX '19.)

So far:

- Easy univariate constructions.
- 3-layer box constructions over $\mathbb{R}^d$: size $\left(^{\mathrm{Lip}}/_{\epsilon}\right)^{\mathcal{O}(d)}$.
- 2-layer RBF convolutions over $\mathbb{R}^d$: size $\left(^{d\cdot\mathrm{Lip}}/_{\epsilon}\right)^{\mathcal{O}(d)}$.

Remarks.

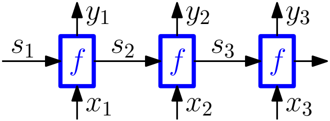
- Impractical constructions! Bad Lipschitz constants.
- Contrast with polynomials: flat pieces.
- Usefuleness of infinite width! Note also:

$$\mathbb{E}\sigma_{\mathrm{r}}(a^{\mathsf{T}}x) = \frac{1}{2}\mathbb{E}|a^{\mathsf{T}}x| = \frac{\|x\|}{\sqrt{2\pi}}.$$

- Poor complexity measures outside univariate!

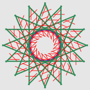
Interlude: three questions

1. Are fixed DN architectures closed under addition?
2. Can RNNs model Turing Machines?



3. Given continuous $g : \mathbb{R}^d \to \mathbb{R}$,
   can we construct custom univariate activations so that

$$g(x) \overset{!}{=} \sum_{i=0}^{2d} f_i \left(\sum_{j=1}^{d} h_{i,j}(x_j) \right)?$$



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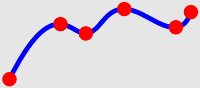

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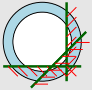

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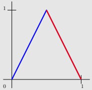

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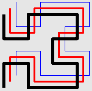
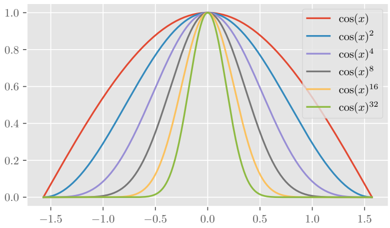


Odds & ends.

Bumps via multiplication



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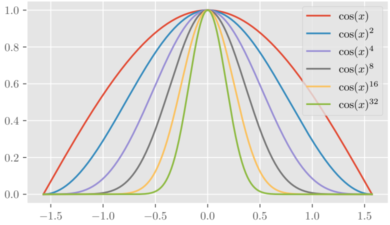

Univariate bump: $\cos(x)^p$ for large p.

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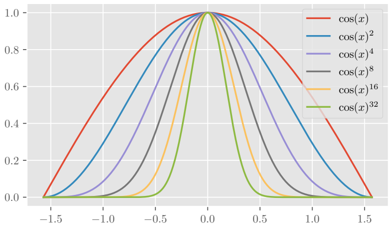
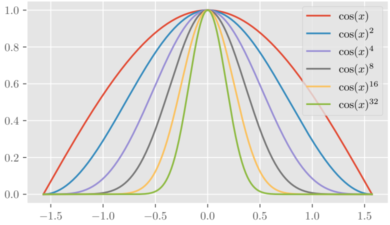


Univariate bump: $\cos(x)^p$ for large p. Multivariate bump:

$$\mathbb{1}\left[\|x\|_\infty \leq 1\right] = \prod_{i=1}^{d} \mathbb{1}\left[|x_i| \leq 1\right] \qquad \text{and} \qquad \prod_{i=1}^{d} \cos(x_i)^p.$$

Bumps via multiplication



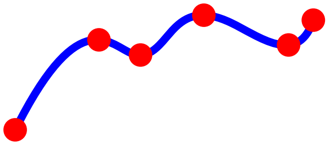
Univariate bump: $\cos(x)^p$ for large p. Multivariate bump:

$$\mathbb{1}\left[\|x\|_\infty \leq 1\right] = \prod_{i=1}^{d} \mathbb{1}\left[|x_i| \leq 1\right] \qquad \text{and} \qquad \prod_{i=1}^{d} \cos(x_i)^p.$$

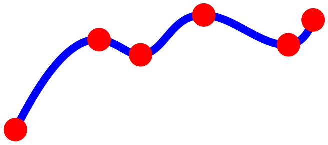
To remove the product:

$$\cos(x)\cos(x) = \frac{1}{2}\left(\cos(2x) + 1\right),$$

$$2\cos(x_1)\cos(x_2) = \cos(x_1 + x_2) + \cos(x_1 - x_2).$$



Theorem (Weierstrass, 1885). Polynomials can uniformly approximate continuous functions over compact sets.



Theorem (Weierstrass, 1885). Polynomials can uniformly approximate continuous functions over compact sets.

Remarks.

- Not a consequence of interpolation: must control behavior between interpolants.
- Proofs are interesting; e.g., Bernstein (Bernstein polynomials and tail bounds), Weierstrass (Gaussian smoothing gives analytic functions). ...
- **Stone-Weierstrass theorem:** Polynomial-like function families (e.g., closed under multiplication) also approximate continuous function.

Theorem (Hornik-Stinchcombe-White '89).
Let $\sigma : \mathbb{R} \to \mathbb{R}$ be given with

$$\lim_{z \to -\infty} \sigma(z) = 0, \qquad \lim_{z \to +\infty} \sigma(z) = 1,$$

and define $\mathcal{H}_\sigma := \left\{ x \mapsto \sigma(a^\mathsf{T} x - b) : (a, b) \in \mathbb{R}^{d+1} \right\}$.
Then $\operatorname{span}(\mathcal{H}_\sigma)$ uniformly approximates
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Then $\operatorname{span}(\mathcal{H}_\sigma)$ uniformly approximates
continuous functions on compact sets.

Proof #1. $\mathcal{H}_{\cos}$ is closed under products since

$$2 \cos(a) \cos(b) = \cos(a + b) + \cos(a - b).$$

Now uniformly approximate fixed $\mathcal{H}_{\cos}$ with $\operatorname{span}(\mathcal{H}_\sigma)$.
(Univariate fitting.)

Proof #2. $\mathcal{H}_{\exp}$ is closed under products since $e^a e^b = e^{a+b}$.
Now uniformly approximate fixed $\mathcal{H}_{\exp}$ with $\operatorname{span}(\mathcal{H}_\sigma)$.
(Univariate fitting.)

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Remarks.

- ReLU is fine: use $\sigma(z) := \sigma_\mathrm{r}(z) - \sigma_\mathrm{r}(z-1)$.
- Size estimate: expanding terms, seem to get $\left({}^{\mathrm{Lip}}/_{\epsilon}\right)^{\Omega(d)}$.
- Best conditions on σ (Leshno-Lin-Pinkus-Schocken '93): theorem holds **iff** σ not a polynomial.
- Inner hint about DN: no need for explicit multiplication?

Other proofs.

- (Cybenko '89.)
  Assume contradictorily you miss some functions.
  By duality, $0 = \int \sigma(a^\intercal x - b)\,\mathrm{d}\mu(x)$
  for some signed measure μ, all (a, b).
  Using Fourier, can show this implies $\mu = 0$...
- (Leshno-Lin-Pinkus-Schocken '93.)
  If σ a polynomial, ...;
  else can (roughly) get derivatives of all orders,
  polynomials of all orders.
- (Barron '93.)
  Consider activation $x \mapsto \exp(ia^\intercal x)$,
  infinite width form $\displaystyle\int \exp(ia^\intercal x)\widetilde{f}(a)\,\mathrm{d}a.$
  Take real part and sample (Maurey) to get $g \in \mathrm{span}(\mathcal{H}_{\cos})$;
  convert to $\mathrm{span}(\mathcal{H}_\sigma)$ as before.
- (Funahashi '89.) Also Fourier, measure-theoretic.

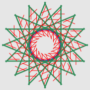
“Universal approximation”

(Uniform approximation of cont. functions on compact sets).

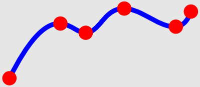
- Elementary proof: RBF (Mhaskar-Michelli ’92; BJTX ’19).
- Slick proof: Stone-Weierstrass and $\mathcal{H}_{\cos}$ or $\mathcal{H}_{\exp}$
  (Hornik-Stinchcombe-White, ’89).
- Proof with size estimates beating $\left({}^{\text{Lip}}/_{\epsilon}\right)^d$,
  indeed norm of Fourier transform of gradient,
  related to “sampling measure”: (Barron ’93).

Remarks.

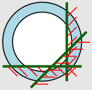
- Exhibits nothing special about DN;
  indeed, same proofs work for boosting, RBF SVM, ...
- Size estimates huge (soon we’ll see $d^{\Omega(d)}$).
- Proofs use nice representation “tricks”;
  (e.g., Leshno et al “iff not polynomial”).



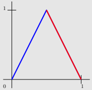
Elementary universal approximation.



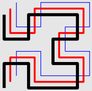
Classical universal approximation.



Benefits of depth.



Sobolev spaces.



Odds & ends.

Radial functions are easy with two ReLU layers

Consider $f(\|x\|^2)$ with Lipschitz constant Lip.

- Pick $h(x) \approx_\epsilon \|x\|_2^2 = \sum_i x_i^2$ with ${}^{d \cdot \mathrm{Lip}}\!/_{\epsilon}$ ReLU.

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- Pick $g \approx_\epsilon f$ with ${}^{\text{Lip}}/_{\epsilon}$ ReLU; then

$$\begin{aligned}\left|f(\|x\|^2) - g(h(x))\right| &\leq \left|f(\|x\|^2) - f(h(x))\right| + \left|f(h(x)) - g(h(x))\right| \\ &\leq \text{Lip}\left|\|x\|^2 - h(x)\right| + \epsilon \leq 2\epsilon.\end{aligned}$$

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Remarks.

- Final size of $g \circ h$ is $\mathrm{poly}(\mathrm{Lip}, d, 1/\epsilon)$.
- Proof style is "typical"/lazy;
  (problematically) pays with Lipschitz constant.
- That was easy/intuitive; how about the 2 layer case?...

Radial functions are ***not*** easy with only one ReLU layer (I)

Theorem (Eldan-Shamir, 2015).

There exists a radial function f,

expressible with two ReLU layers of width $\mathrm{poly}(d)$,

and a probability measure P

so that every g with a single ReLU layer of width $2^{\mathcal{O}(d)}$ satisfies

$$\int \big(f(x) - g(x)\big)^2 \, \mathrm{d}P(x) \geq \Omega(1).$$

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Proof hints.
Apply Fourier isometry and consider the transforms.
Transform of g is supported on a small set of tubes;
transform of f has large mass they can't reach.

Radial functions are ***not*** easy with only one ReLU layer (II)

Theorem (Daniely, 2017).
Let $(x, x') \sim P$ be uniform on two sphere surfaces,
define $h(x, x') = \sin(\pi d^3 x^\top x')$.
For any g with a single ReLU layer
of width $d^{\mathcal{O}(d)}$ and weight magnitude $\mathcal{O}(2^d)$,

$$\int \big(h(x, x') - g(x, x')\big)^2 \, \mathrm{d}P(x, x') \geq \Omega(1),$$

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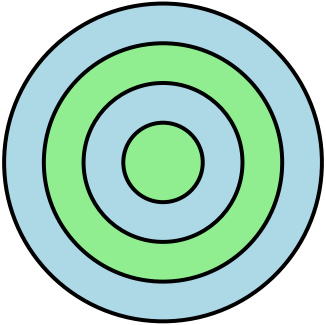
Proof hints.

Spherical harmonics reduce this to a univariate problem;
apply region counting.

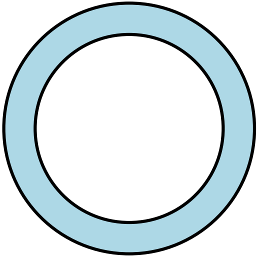
Approximation of high-dimensional radial functions



(A radial function contour plot.)

If we can approximate each shell,
we can approximate the overall function.

Approximation of high-dimensional radial shell



Let's approximate a single shell; consider

$$x \mapsto \mathbb{1}\left[\|x\| \in [1 - {}^{1}\!/\!{}_{d}\ ,\ 1]\right],$$

which has a constant fraction of sphere volume.

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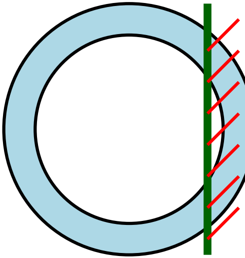

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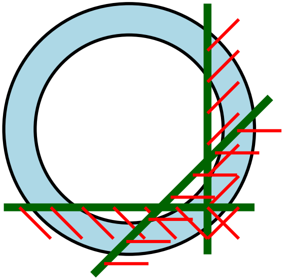

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...but then we need to cover exponentially many caps.

Let's go back to the drawing board;
what do shallow representations do exceptionally badly?

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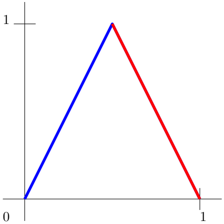
what do shallow representations do exceptionally badly?

One weakness: their complexity scales with #bumps.

Consider the **tent map**

$$\Delta(x) := \sigma_{\mathrm{r}}(2x) - \sigma_{\mathrm{r}}(4x-2) = \begin{cases} 2x & x \in [0, 1/2), \\ 2(1-x) & x \in [1/2, 1]. \end{cases}$$



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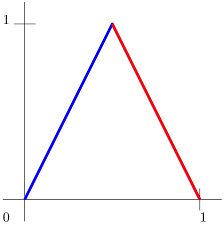

Δ.

What is the effect of composition?

$$f(\Delta(x)) = \begin{cases} x \in [0, {}^1\!/\!_2) & \Longrightarrow & f(2x) = f \text{ squeezed into } [0, 1/2], \\ x \in [{}^1\!/\!_2, 1] & \Longrightarrow & f\bigl(2(1-x)\bigr) = f \text{ reversed, squeezed.} \end{cases}$$

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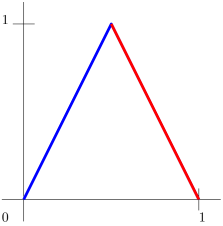

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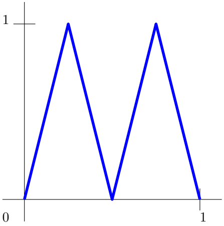

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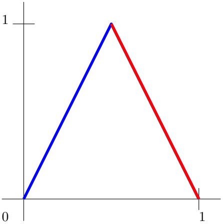
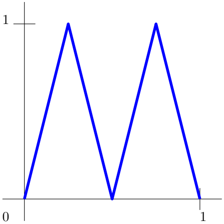


$\Delta.$



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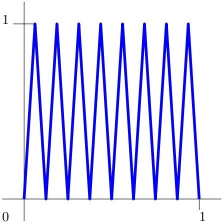

$\Delta^k.$

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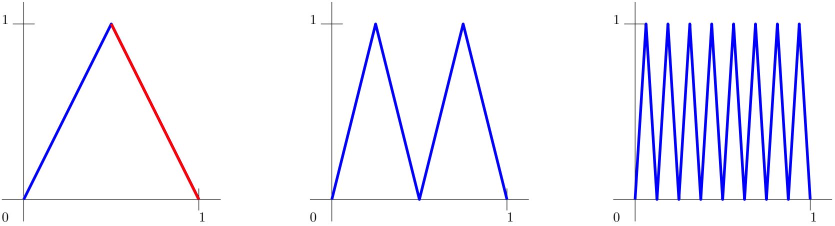

$\Delta.$ $\qquad$ $\Delta^2 = \Delta \circ \Delta.$ $\qquad$ $\Delta^k.$

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$\boxed{\Delta^k \text{ uses } \mathcal{O}(k) \text{ layers \& nodes, but has } \mathcal{O}(2^k) \text{ bumps.}}$

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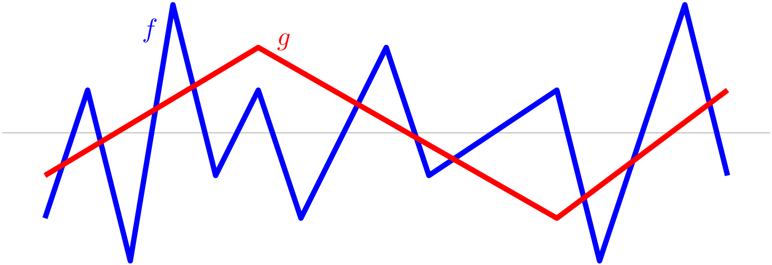
Proof.

1. g with few oscillations can't apx oscillatory regular f.
2. There exists a regular, oscillatory f. ($f = \Delta^{k^2+3}$.)
3. Width m depth $L \implies$ few ($\mathcal{O}(m^L)$) oscillations.
   Rediscovered many times;
   (T '15) gives elementary univariate argument;
   multivariate arguments in (Warren '68), (Arnold ?),
   (Montufar, Pascanu, Cho, Bengio, '14), (BT '18), ...

g with few oscillations;
f highly oscillatory $\implies \int_{[0,1]} |g - f|$ large .



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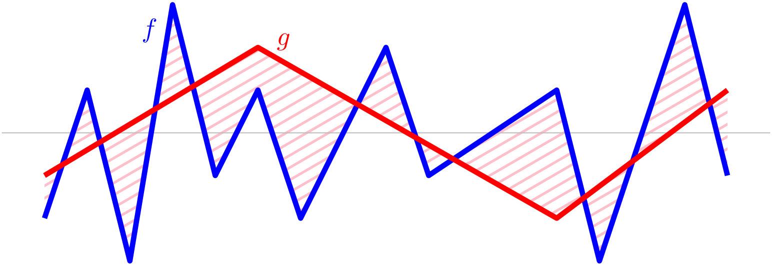
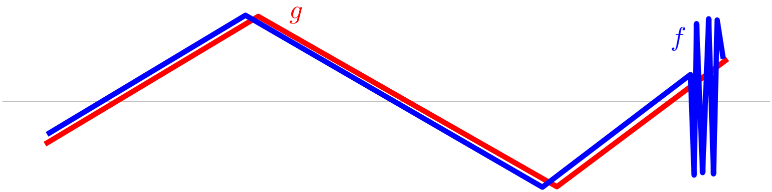


$\color{red}{g}$ with few oscillations;
$\color{blue}{f}$ highly oscillatory $\overset{?}{\Longrightarrow} \displaystyle\int_{[0,1]} |\color{red}{g} - \color{blue}{f}| \text{ large}$.



g with few oscillations;
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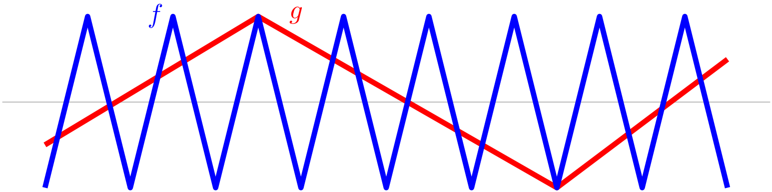

Let's use $f = \Delta^{k^2+3}$.

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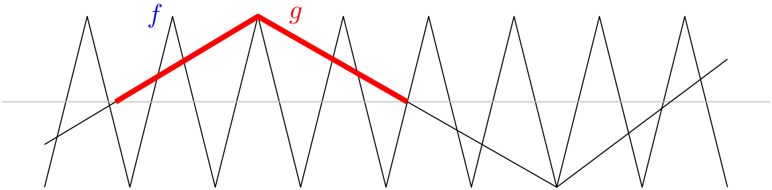

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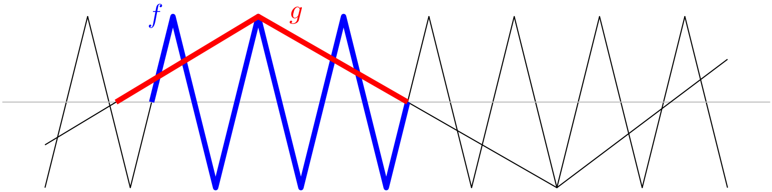

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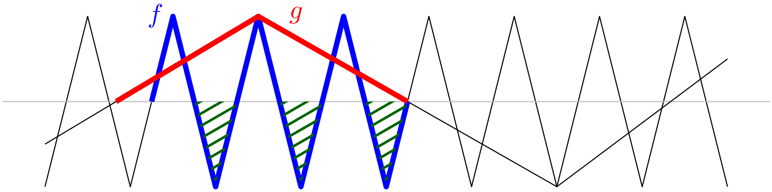

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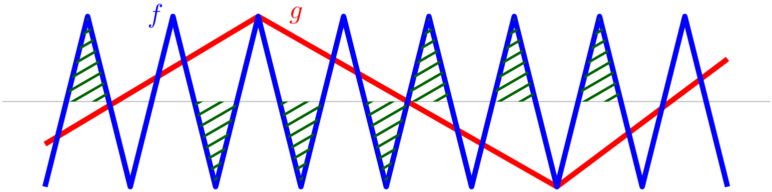

Let's use $f = \Delta^{k^2+3}$.

Story from *benefits of depth*:

- Certain radial functions have
  polynomial width 2 ReLU layer representation,
  exponential width 1 ReLU layer representation.
- Δ^{k^2+3} can be written with $\mathcal{O}(k^2)$ depth and $\mathcal{O}(1)$ width, requires width $\Omega(2^k)$ if depth $\mathcal{O}(k)$.

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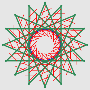
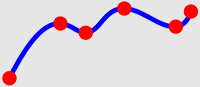
  requires width $\Omega(2^k)$ if depth $\mathcal{O}(k)$.

Remarks.

- Δ^k is 2^k-Lipschitz;
  possibly nonsensical, unrealistic.
- These results have stood a few years now;
  many "technical" questions,
  also "realistic" questions.



Elementary universal approximation.



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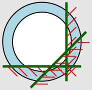

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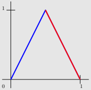
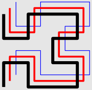
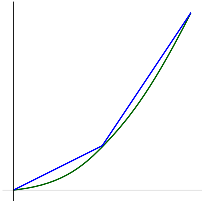


Sobolev spaces.



Odds & ends.

$$h_k := \text{piecewise-affine interpolation of } x^2 \text{ at } \{0, \tfrac{1}{2^k}, \tfrac{2}{2^k}, \ldots, \tfrac{2^k}{2^k}\}.$$



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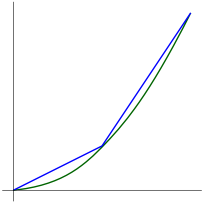

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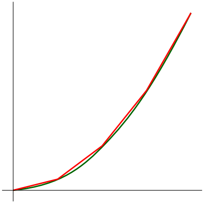

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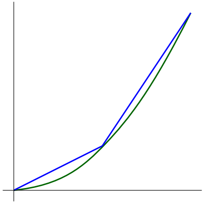

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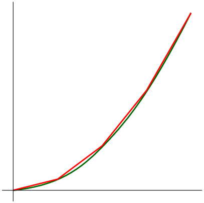

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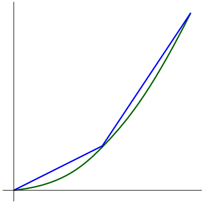

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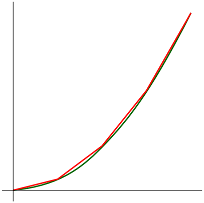

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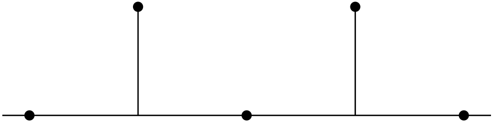

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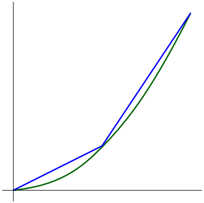

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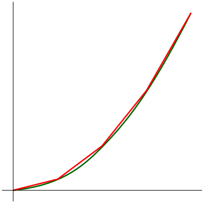

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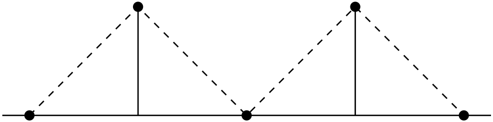

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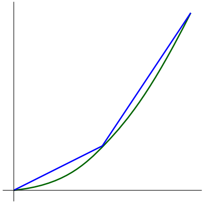

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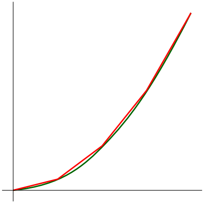

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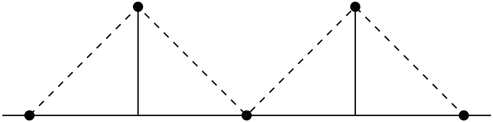

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Thus $h_k(x) = x - \sum_{i \leq k} \Delta^i(x)/4^i$.

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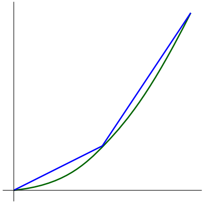

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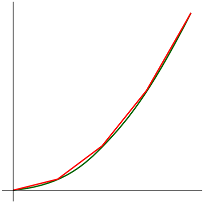

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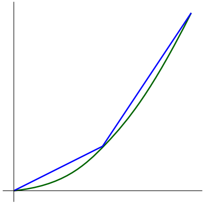

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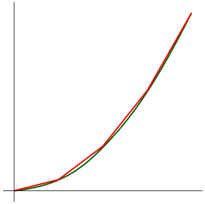

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Thus $h_k(x) = x - \sum_{i \leq k} \Delta^i(x)/4^i$.

- h_k needs $k = \mathcal{O}(\ln(1/\epsilon))$ to ϵ-apx $x \mapsto x^2$ (Yarotsky, '16), with matching lower bounds.
- Squaring implies **multiplication** via polarization:
  $x^\mathsf{T} y = \dfrac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$.

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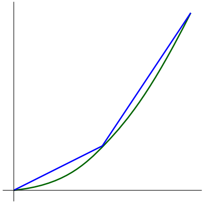

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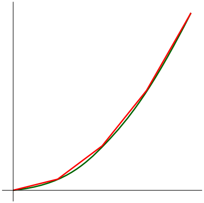

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- Squaring implies **multiplication** via polarization:
  $$x^\top y = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right).$$
- This implies efficient approximation of polynomials; can we do more?

Theorem (Yarotsky '16).

Let dimension d and smoothness order r be given. Given $f : [0,1]^d \to \mathbb{R}$, all rth order derivatives bounded by 1, exists a network g

with $C_{d,r} \ln(e/\epsilon)$ layers and $C_{d,r}\epsilon^{-d/r} \ln(e/\epsilon)$ nodes

so that

$$\sup_{x\in[0,1]^d} |f(x) - g(x)| \le \epsilon.$$

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Proof.

Conditions imply accurate local Taylor expansions.

Therefore can write f as a linear combination of this basis:

polynomials multiplied by local bumps.

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Remarks.

- There is depth, but it is function independent: only the basis coefficients use f.
- This is a shallow representation: only the basis coefficients
- Lipschitz constant is possibly bad: $\Delta^{1/\epsilon}$ is $1/\epsilon$-Lipschitz, the bumps are $1/\epsilon^{d/r}$-Lipschitz.

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Remarks.

- There is parallel and subsequent work with similar proof ideas and Lipschitz constants: (Safran-Shamir '16), (Petersen-Voigtlaender '17), (Schmidt-Hieber '17).
- Another appearance of polynomials in DN: **Sum-product networks**. These were the first to have depth separation (Delalleau-Bengio '11).

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Remarks.

- DN can approximate polynomials efficiently,
  but the reverse is false:
  a single ReLU requires degree $1/\epsilon$.
- Polynomials can not handle flat regions well;
  this is used above,
  and in approximated rational functions (T '17).

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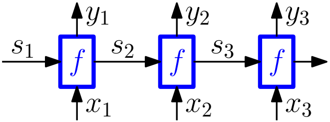
Remarks.

- Corresponding lower bounds indicate depth is needed.

Interlude: three questions

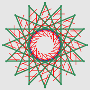
1. Are fixed DN architectures closed under addition?
   No, add together perturbed copies of Δ^k.
2. Can RNNs model Turing Machines?



   Hint. ReLU networks can do exact Boolean formulae.
   Set f to state transition table,
   encode tape on s.
3. Given continuous $g : \mathbb{R}^d \to \mathbb{R}$,
   can we construct custom univariate activations so that

$$g(x) \overset{!}{=} \sum_{i=0}^{2d} f_i \left(\sum_{j=1}^{d} h_{i,j}(x_j) \right) ?$$

   Hint? Contradicts a Hilbert problem?



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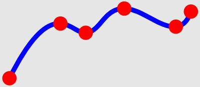

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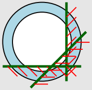

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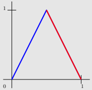

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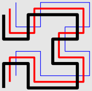

Odds & ends.

Generative modeling

Typical setup: *pushforward measure* $g\#\mu$, meaning

$$\text{sample } x \sim \mu, \qquad \text{output } g(x).$$

Many easy constructions have bad/∞ Lipschitz constants!
E.g., mapping uniform into $[0, 1/2]$, $(3/2, 2]$.

Some literature:
(Arora-Ge-Liang-Ma-Zhang '17, BT '18, Bai-Ma-Risteski '19, Elchanan's talk this week!)

Randomly *initialized* networks

Approximation fact in recent optimization papers:
a small perturbation of random initialization
gives any function you want!
(Du-Lee-Li-Wang-Zhai '18, AllenZhu-Li-Song '18).

There is residual error from the noise
approximating high-Lipschitz functions is problematic!
(BJTX '19.)

Randomly *sampled* networks

Theorem. With probability $\geq 1 - {}^1\!/_e$,

$$\sup_{\|x\|_2 \leq 1} \left| \int \sigma_{\mathrm{r}}(a^{\mathsf{T}} x - b) \, \mathrm{d}\mu(a, b) - \frac{\|\mu\|_1}{N} \sum_{i=1}^{N} \sigma_{\mathrm{r}}(a_i^{\mathsf{T}} x - b_i) \right| \leq \mathcal{O}\left(\frac{B \|\mu\|_1}{\sqrt{N}} \right),$$

where support of μ has $\|(a, b)\| \leq B$.

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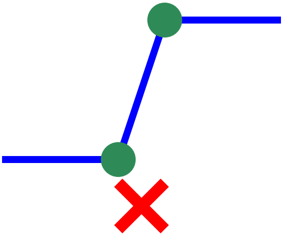
where support of μ has $\|(a,b)\| \leq B$.

Proof. Invoke Rademacher complexity,
but swap inputs and parameters.

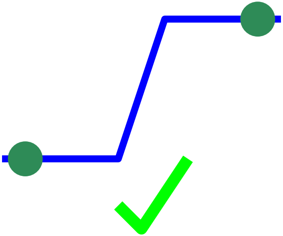
(Koiran-Gurvits '97, Sun-Gilbert-Tewari '18, BJTX '19.)
Also Maurey's Lemma (Barron '93).

Adversarial stability



Adversarial examples lower bound the Lipschitz constant...

Adversarial stability



Adversarial examples lower bound the Lipschitz constant…

…but a bad Lipschitz constant
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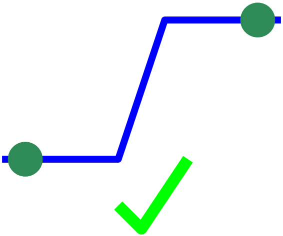

Adversarial examples lower bound the Lipschitz constant…

…but a bad Lipschitz constant
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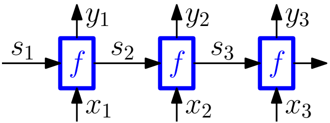
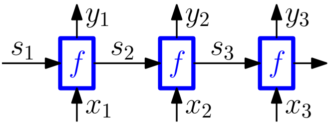
Given the existence of adversarial examples,
uniform approximation too stringent?

Turing machines and RNNs



- Make f the TM state transition table,
  s the tape.

Turing machines and RNNs



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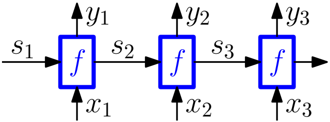

- Make f the TM state transition table, s the tape.
- $x \mapsto \mathbb{1}[x \geq 0]$ is **not computable**; bits need a special encoding within s.
- Use a **robust "cantor-like" encoding**.

(Siegelmann-Sontag '94.)

Kolmogorov-Arnold '56

There exist continuous $((h_{i,j})_{i=0}^{2d})_{j=1}^{d} : \mathbb{R} \to \mathbb{R}$,
so that for any continuous $g : \mathbb{R}^d \to \mathbb{R}$,
there exist continuous $(f_i)_{i=0}^{2d} : \mathbb{R} \to \mathbb{R}$
with

$$g(x) = \sum_{i=0}^{2d} f_i\Big(\sum_{j=1}^{d} h_{i,j}(x_j)\Big).$$

Kolmogorov-Arnold '56

There exist continuous $((h_{i,j})_{i=0}^{2d})_{j=1}^{d} : \mathbb{R} \to \mathbb{R}$,
so that for any continuous $g : \mathbb{R}^d \to \mathbb{R}$,
there exist continuous $(f_i)_{i=0}^{2d} : \mathbb{R} \to \mathbb{R}$
with

$$g(x) = \sum_{i=0}^{2d} f_i\Big(\sum_{j=1}^{d} h_{i,j}(x_j)\Big).$$

Step 1.
Fix target accuracy $\epsilon > 0$.

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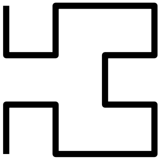
Step 2.
Choose $f : \mathbb{R} \to \mathbb{R}$,
nearly injective $Q : \mathbb{R}^d \to \mathbb{R}$,
$g \approx f(Q(x))$



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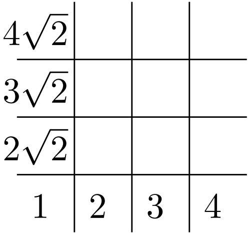
Step 3.
Replace near-injection $Q : \mathbb{R}^d \to \mathbb{R}$
with $\sum_j h_j(x_j)$.



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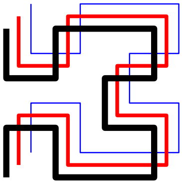
Step 4.
Replace $f(\sum_j h_j(x_j))$
with staggered versions $\sum_i f_i(\sum_j h_{i,j}(x_j))$;
for any $x \in [0,1]^d$,
$\geq$ half are correct.



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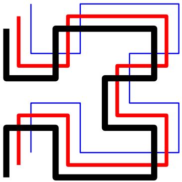
Step 5.
Embed the solutions for infinitely many ϵ into one.

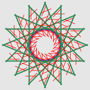


Main story.

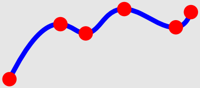
- Can fit continuous functions in various ways;
  the size is bad ($\left(d \cdot \mathrm{Lip}/\epsilon\right)^{\mathcal{O}(d)}$).
- Composition and depth bring some concrete benefits;
  exponential reductions in width!
- Polynomials may be efficiently approximated,
  but also some non-polynomials
  (Sobolov balls, rational functions, flat regions, ...).

Remarks.

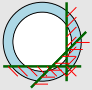
- Refined depth separations (e.g., a single new layer)
  and practical depth separations
  are still elusive.
- Refined, average-case complexity measures are elusive.



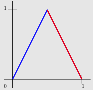
Elementary universal approximation.



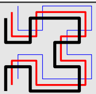
Classical universal approximation.



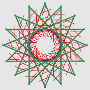
Benefits of depth.



Sobolev spaces.



Odds & ends.



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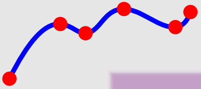

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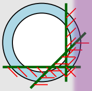
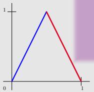


Thanks. . . any questions?



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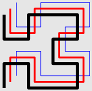

Odds & ends.