# Thin Trees and Interlacing Families on Strongly Rayleigh Distributions


Nima Anari

Stanford University / Simons Institute for the Theory of Computing


based on joint work with

Shayan Oveis Gharan

# Brief Intro to Interlacing Families

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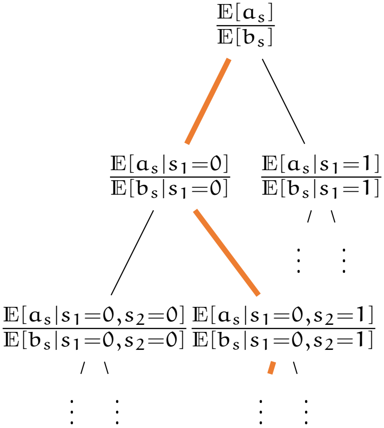

- Polynomials: Let $p_s(x) = b_s x - a_s$. Then $\text{root}(p_s) = \frac{a_s}{b_s}$ and $\text{root}(\mathbb{E}[p_s]) = \frac{\mathbb{E}[a_s]}{\mathbb{E}[b_s]}$.

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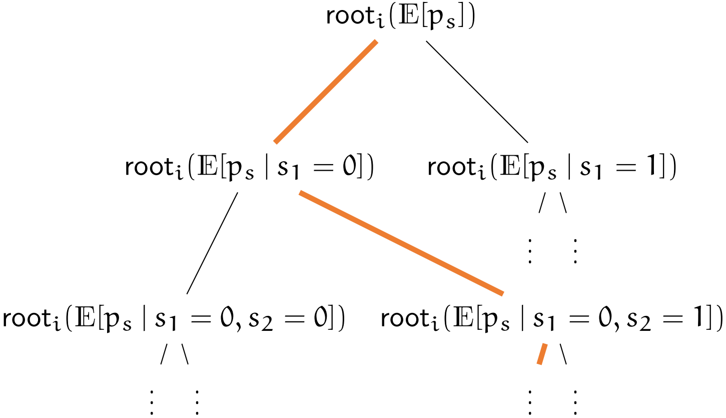

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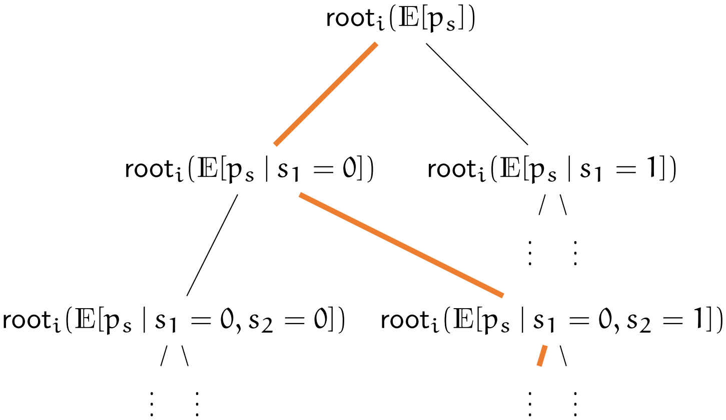



Works as long as all nodes are real-rooted and so are all convex combinations of siblings.

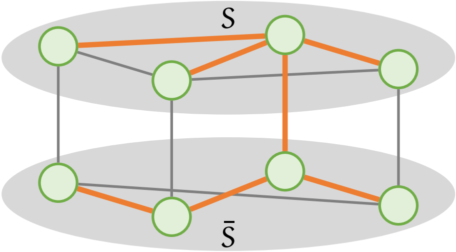
# Thin Tree and Spectrally Thin Tree

**Thinness**

T is $\alpha$-thin w.r.t. G iff

$$|T(S, \bar{S})| \leqslant \alpha \cdot |G(S, \bar{S})|,$$

for every subset of vertices $S$.



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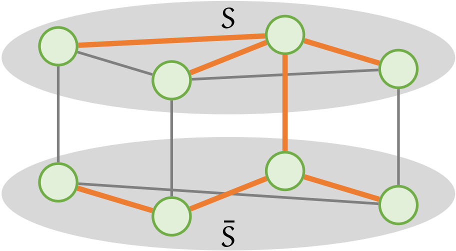

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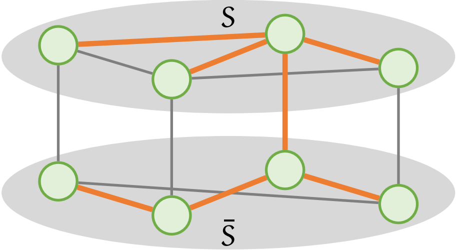


$\alpha$-spectrally thin
$\Longrightarrow$ $\alpha$-thin
[on board ...]

# Structure of the Talk

1. Thin Trees
   - Random Spanning Trees
   - Statement Needed from Interlacing Families
   - Well-Conditioning
2. Interlacing Families on Strongly Rayleigh Distributions
   - Statement Needed from Interlacing Families
   - Proof Sketch

# Thin Tree Conjecture

**Strong Form of [Goddyn]**

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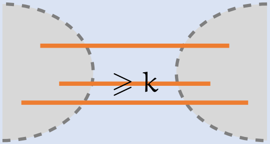
**[A-Oveis Gharan'15]**

There is always a $\log\log^{O(1)}(n)/k$-thin tree.

# Spectral Thinness

**Edge Connectivity**

$$|G(S, \bar{S})| \geqslant k$$




Goal

**Thin Tree**

$$|T(S, \bar{S})| \leqslant \alpha \cdot |G(S, \bar{S})|$$

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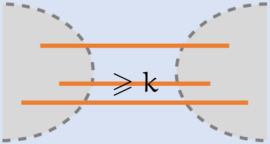


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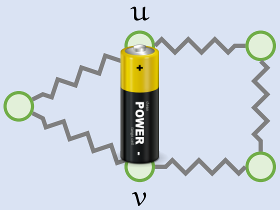
## Electrical Connectivity

$$\text{Reff}(u, v) \leqslant \frac{1}{k}$$

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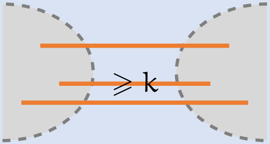


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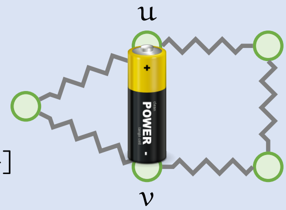


[Harvey-Olver'14, Marcus-Spielman-Srivastava'14]

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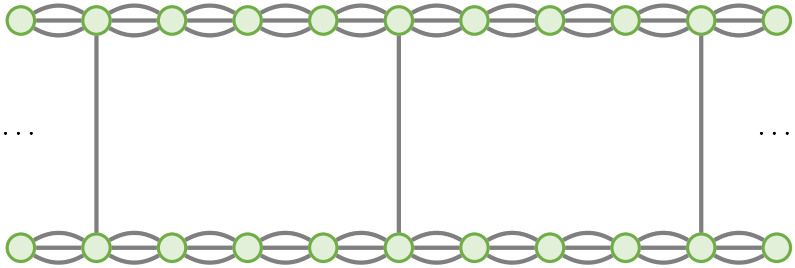
# Obstacles

- Problem: Edge connectivity does not imply electrical connectivity.



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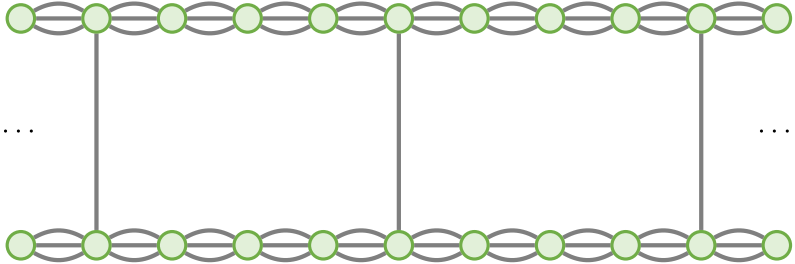

- Problem: Electrical connectivity is needed for the existence of spectrally thin trees. For any $e = (u, v) \in T$:

$$1 \geqslant \mathrm{Reff}_T(u, v) = e^\intercal L_T^- b_e \geqslant \frac{1}{\alpha} \cdot b_e^\intercal L_G^- b_e = \frac{1}{\alpha} \cdot \mathrm{Reff}_G(u, v).$$

Key Idea : Well-condition the graph spectrally without changing cuts much.

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- Add “graph” $H$ to $G$ ensuring

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- Goal: Find H that brings Reff down.
- Problem 1: How do we ensure T does not use any newly added edges?
- Problem 2: How do we certify H is $O(1)$-thin w.r.t. G?

Ensuring only original edges are picked ...

# Interlacing Families on Strongly Rayleigh Distributions

**Corollary of** [Marcus-Spielman-Srivastava'14, Harvey-Olver'14]

If for every edge $e$ in a graph $G$

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Let $F$ be a subset of edges in $G$. If for every $e \in F$,

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[on board . . . ]

# Ensuring cuts do not blow up . . .

# Idea 1: Using Shortcuts

- If $H$ can be routed over $G$ with congestion $O(1)$, then for every $S$

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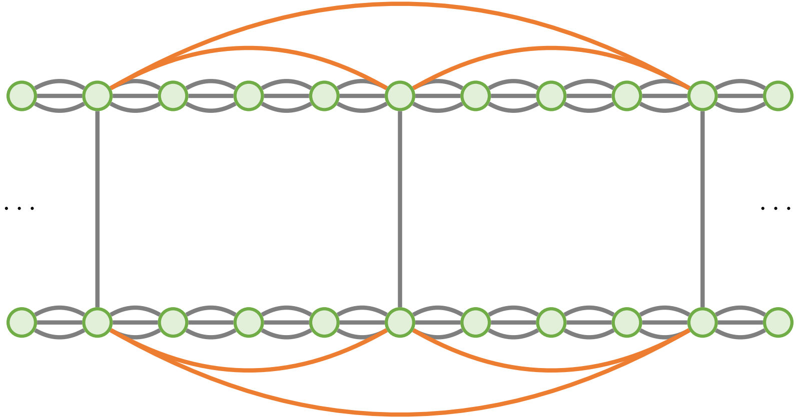

# Idea 2: Check All Constraints

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$$\min_{D \succeq 0} \left\{ \max_{e \in G} \mathrm{Reff}_D(e) \;\middle|\; \forall S : \mathbb{1}_S^\intercal D \mathbb{1}_S \leqslant \mathbb{1}_S^\intercal L_G \mathbb{1}_S \right\}$$

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- Pro: Can use duality to facilitate analysis.
- Con: Adds another obstacle to making the construction algorithmic.

# Puzzle Interlude: Degree-thinness . . .

## Degree-Thin Trees (Toy Example)

Suppose that we want a tree which is thin only in degree cuts, i.e.,

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- Is there an easy well-conditioner $H$?
- An expander!

[on board ...]

Do well-conditioners always exist?

- What is the worst possible answer to the convex program?

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**Averages in Degree Cuts [A-Oveis Gharan'15]**

For every $k$-edge-connected graph $G$ there is a $1$-thin matrix $D \succeq 0$ such that for every singleton $S$

$$\mathbb{E}[\mathrm{Reff}_D(e) \mid e \in G(S, \bar{S})] \leqslant \frac{(\log \log n)^{O(1)}}{k}.$$

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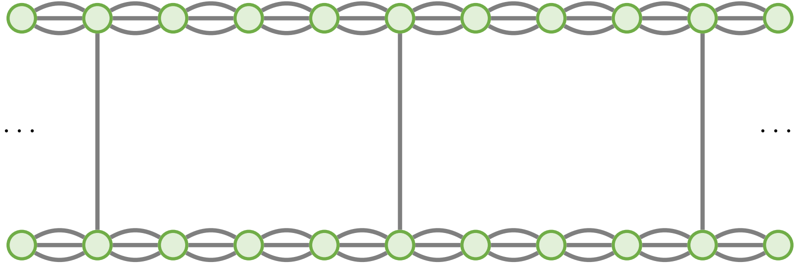
**Informal Lemma**

Every graph has weakly expanding induced subgraphs.

Plan: Contract this subgraph, and repeat to get a hierarchical decomposition.
Lower average **Reff** in degree cuts of each expander simultaneously.

# Example: Planar Graphs

If $G$ is planar, there are vertices $u$ and $v$ connected by $\Omega(k)$ edges.



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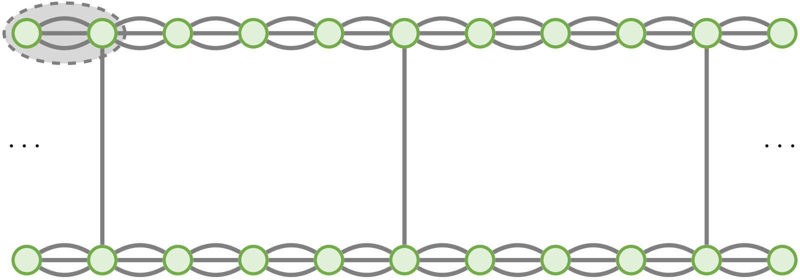
If $G$ is planar, there are vertices $u$ and $v$ connected by $\Omega(k)$ edges.



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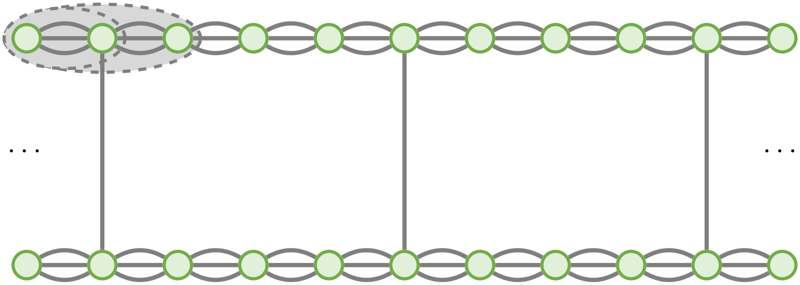

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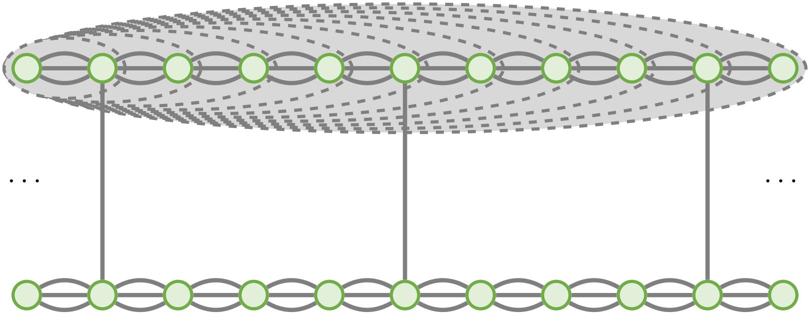

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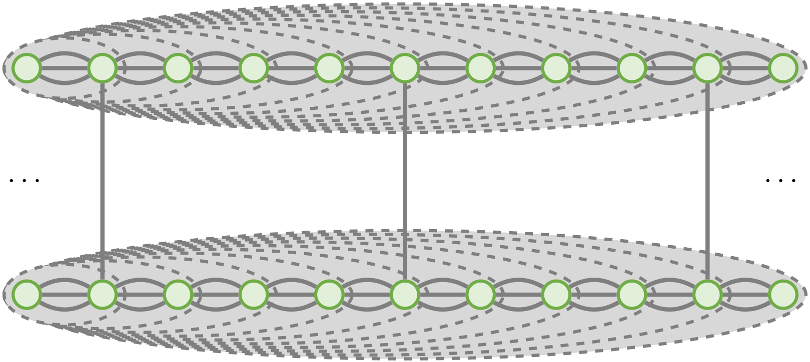

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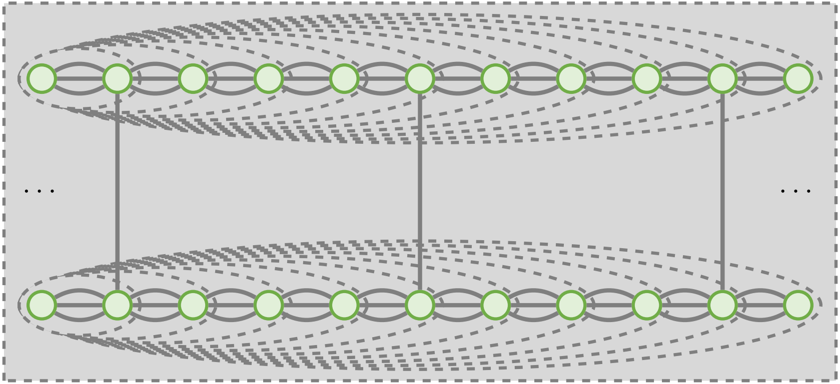

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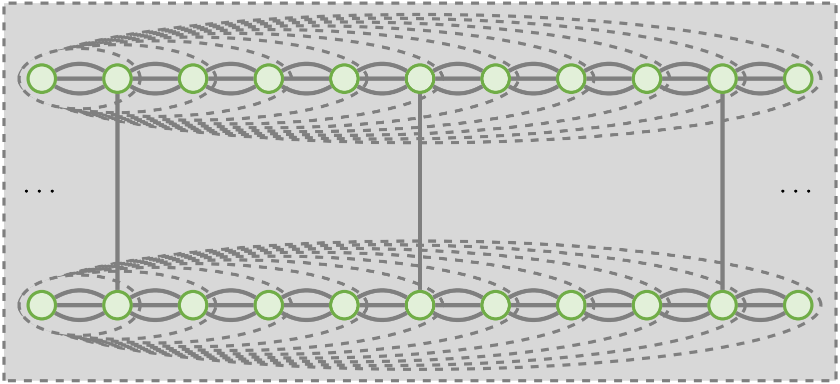

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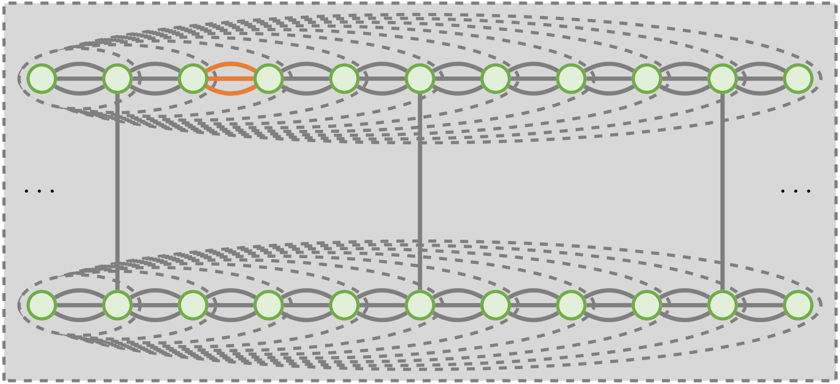


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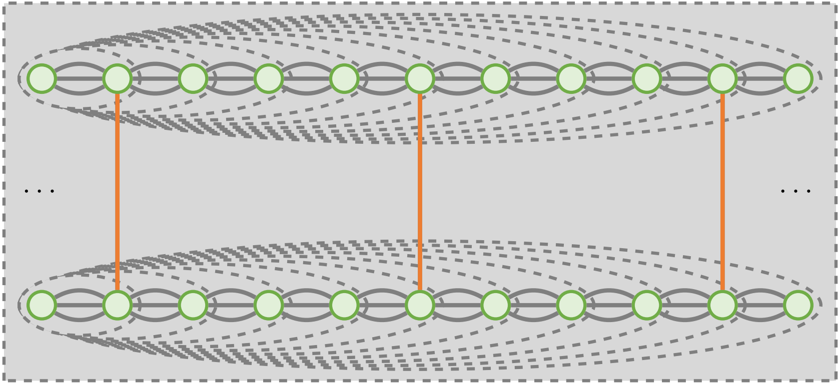


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- Repeat this $\log\log n$ times until expansion is $\Omega(1)$.

# Structure of the Talk

1. Thin Trees
   - Random Spanning Trees
   - Statement Needed from Interlacing Families
   - Well-Conditioning
2. Interlacing Families on Strongly Rayleigh Distributions
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   - Proof Sketch

# Goal Statement

If $L_1, \ldots, L_m \succeq 0$ are rank 1 and $\mu : \binom{[m]}{d} \to \mathbb{R}_{\geqslant 0}$ is Strongly Rayleigh then

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[on board ...]

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Thank you!