Geometry of Polynomials Bootcamp **Winter 2019**

Completely Log-Concave Polynomials and Distributions

Speaker: Nima Anari *Exercises*

(P1) **Homogenization of Completely Log-Concave Polynomials**

Suppose that $f(z_1, \ldots, z_n) = f_0 + f_1 + \cdots + f_d$, where f_k is a k-homogeneous polynomial in $z_1, \ldots, z_n$. Prove that f is completely log-concave if and only if the following polynomial in the variables y and $z_1, \ldots, z_n$ is completely log-concave:

$$\frac{y^d}{d!} f_0 + \frac{y^{d-1}}{(d-1)!} f_1 + \cdots + \frac{y^0}{0!} f_d.$$

(P2) **Universality of Log-Concavity for Homogeneous Polynomials**

Suppose that $f(z_1, \ldots, z_n)$ is a d-homogeneous polynomial with nonnegative coefficients. Prove that the following are equivalent:

(a) f is log-concave over $\mathbb{R}^n_{\geq 0}$.

(b) $f^{1/d}$ is concave over $\mathbb{R}^n_{\geq 0}$.

(c) f is quasi-concave over $\mathbb{R}^n_{\geq 0}$, that is $f^{-1}([1, \infty)) \cap \mathbb{R}^n_{\geq 0}$ is a convex set.

(P3) **Univariate and Multiaffine Bivariate Polynomials**

Prove that the polynomial $a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d$ is completely log-concave if and only if $0! \cdot a_0, 1! \cdot a_1, \ldots, d! \cdot a_d$ is a log-concave sequence.

For what a, b, c, d is the polynomial $a + bz_1 + cz_2 + dz_1 z_2 \in \mathbb{R}[z_1, z_2]$ completely log-concave?

Prove that if $S \subseteq [n]$ is a random subset whose distribution is completely log-concave, then for every pair of distinct elements $i, j \in [n]$

$$\mathbb{P}[i, j \in S] \leq 2\mathbb{P}[i \in S]\mathbb{P}[j \in S].$$

(P4) **Coefficient Products**

Suppose that $f, g \in \mathbb{R}[z_1, \ldots, z_n]$ are completely log-concave polynomials. Let h be defined such that for every $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$

$$\partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} h\big|_{z=0} = \left(\partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} f\big|_{z=0}\right) \cdot \left(\partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} g\big|_{z=0}\right).$$

Prove that h is completely log-concave in the following cases: $n = 1$, or when $n = 2$ and f, g are homogeneous. Give an example outside of these cases where h is not completely log-concave.

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(P1) **Products**

Prove that when $f, g \in \mathbb{R}[z_1, \ldots, z_n]$ are completely log-concave, then so is $f \cdot g$.

(P2) **Operators Preserving Complete Log-Concavity**

Prove that if $T : \mathbb{R}[z_1, \ldots, z_n]_{\leq 1} \to \mathbb{R}[z_1, \ldots, z_n]$ is a linear map that preserves real-stability and maps polynomials with nonnegative coefficients to polynomials with nonnegative coefficients, then T preserves complete log-concavity.

Hint: Prove that if the symbol of T is completely log-concave, then T preserves complete log-concavity.