Computing Partition Functions by Polynomial Interpolation

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Lectures on Friday, January 25, 9:30 a.m.–10:50 a.m. and 11:15 a.m. – 12:35 p.m.

In-class exercises:

1. Problem. For any $0 < \delta < 0.5$, construct a polynomial $\phi = \phi_\delta : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\phi(0) = 0$, $\phi(1) = 1$, $\deg \phi = e^{O(1/\delta)}$ and ϕ maps the disc

$$\mathbb{D} = \mathbb{D}_\beta = \{z \in \mathbb{C} : \ |z| < \beta\} \quad \text{of some radius} \quad \beta = 1 + \frac{1}{e^{O(1/\delta)}}$$

inside the rectangle

$$-\delta \ \leq \ \Re z \ \leq \ 1 + \delta \quad \text{and} \quad |\Im z| \ \leq \ \delta$$

(all implied constants in the "O" notation are absolute).

Hint: For $\rho > 0$, consider the function

$$f(z) = f_\rho(z) = \rho \ln \frac{1}{1 - \alpha z} \quad \text{where} \quad \alpha = \alpha_\rho = 1 - e^{-\frac{1}{\rho}}$$

(we consider the principal branch of the logarithm).

2. Problem. For $0 < \delta < 1$, consider the rational function $\psi = \psi_\delta : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ defined by

$$\psi(z) = \frac{\delta}{(1 - \xi z)^2} - \delta \quad \text{where} \quad \xi = \xi_\delta = 1 - \sqrt{\frac{\delta}{1 + \delta}}.$$

Show that $\psi(0) = 0$, $\psi(1) = 1$ and that the image $\psi(\mathbb{D})$ of the disc

$$\mathbb{D} = \mathbb{D}_\beta = \{z \in \mathbb{C} : \ |z| < \beta\} \quad \text{of radius} \quad \beta = 1 + \sqrt{\delta}$$

does not intersect the ray

$$\{z \in \mathbb{C} : \ \Im z = 0 \quad \text{and} \quad \Re z \ \leq \ -\delta\}.$$

3. Problem. Let $G = (V, E)$ be an undirected graph with set V of vertices, set E of edges, without loops or multiple edges. A map $\phi : V \longrightarrow \{1, \ldots, q\}$ is called a *proper q-coloring* if $\phi(v) \neq \phi(u)$ whenever $\{u, v\} \in E$. Let $\chi_G(q)$ be the number of proper q-colorings. Prove that

$$\chi_G(q) = \sum_{E' \subset E} q^{c(E')} (-1)^{|E'|},$$

where $c'(E')$ is the number of connected components of the graph $G' = (V, E')$.

Hint: Write

$$\chi_G(q) = \sum_{\phi : V \longrightarrow \{1, \ldots, q\}} \ \prod_{\{u,v\} \in E} \left(1 - \delta_{\phi(u)\phi(v)}\right) \quad \text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Homework problems:

1. Problem. Let

$$\mathbb{D} = \mathbb{D}_\beta = \{z \in \mathbb{C} : \ |z| < \beta\}$$

be the disc of some radius $\beta > 1$ and let $\phi : \mathbb{D} \longrightarrow \mathbb{C}$ be a holomorphic function such that $\phi(0) = 0$, $\phi(1) = 1$ and ϕ maps $\mathbb{D}$ inside the strip

$$-\delta \ < \ \Re\, z \ < \ 1 + \delta \quad \text{and} \quad |\Im\, z| \ < \ \delta$$

for some $0 < \delta < 0.5$. Prove that

$$\beta \ < \ 1 + c_1 e^{-c_2/\delta} \quad \text{for some absolute constants} \quad c_1, c_2 > 0.$$

Hint: Use the Schwarz Lemma.

2. Problem. Let $A = (a_{ij})$ be an $n \times n$ complex matrix such that

$$\sum_{j=1}^{n} |a_{ij}| \ < \ 1 \quad \text{for} \quad i = 1, \ldots, n.$$

Show that $\operatorname{per}(I + A) \neq 0$, where I is the $n \times n$ identity matrix.

Hint: Argue that it suffices to assume that $a_{ii} = 0$ for $i = 1, \ldots, n$ and that every row of A contains at most one non-zero entry.

3. Problem. Prove that for any $0 < \delta < 1$ there exists an $\epsilon = \epsilon(\delta) > 0$ such that if $A = (a_{ij})$ is $n \times n$ complex matrix such that

$$\delta \ \leq \ \Re\, a_{ij} \ \leq \ 1 \quad \text{and} \quad |\Im\, a_{ij}| \ \leq \ \epsilon \quad \text{for all} \quad i, j$$

then $\operatorname{per} A \neq 0$.

Hint: Show by induction on n that for an appropriate $\epsilon > 0$ a stronger statement holds: if A and B are $n \times n$ complex matrices as above that differ in at most one row (at most one column) then the angle between the non-zero complex numbers $\operatorname{per} A \neq 0$ and $\operatorname{per} B \neq 0$, considered as vectors in $\mathbb{R}^2 = \mathbb{C}$, does not exceed $\pi/2$.