Proof Complexity Lower Bounds from Graph Expansion and Combinatorial Games

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Based on joint work with Massimo Lauria and Mladen Mikša

Proof Complexity and Expansion

- **General goal:** Prove that concrete proof systems cannot efficiently certify unsatisfiability of concrete CNF formulas

- **General theme:**

CNF formula $\mathcal{F}$ "expanding"

$\Downarrow$

Large proofs needed to refute $\mathcal{F}$

- Paradigm implemented for
  - resolution: well-developed machinery
  - polynomial calculus: very much less so

  (Will define these proof systems shortly)

- What "expanding" means is usually a formula-specific hack

Lower Bounds by Playing Games on Graphs

Given CNF formula $\mathcal{F}$ over variables $\mathcal{V}$, build bipartite graph

- Left vertex set partition of clauses into $\mathcal{F} = \bigcup_{i=1}^{m} F_i$
- Right vertex set division of variables $\mathcal{V} = \bigcup_{j=1}^{n} V_j$
- Edge (F_i, V_j) if $Vars(F_i) \cap V_j \neq \emptyset$

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Lower bound on proof size if

1. Bipartite graph is an expander (very well-connected)
2. We can win the edge game on every edge (F_i, V_j)

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Edge game on (F_i, V_j)

- Adversary assigns all variables $\mathcal{V} \setminus V_j$
- We assign V_j
- We win if F_i true

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- Adversary has to start $\Rightarrow$ resolution lower bound
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Consequences

- Extends techniques in [BW01] and [AR03]
- Unifies many previous lower bounds
- And yields some new ones

Outline

Just To Make Sure We're on the Same Page...

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $\mathcal{F} = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  $k = \mathcal{O}(1)$ constant in this talk
- $\mathit{true} = 1$; $\mathit{false} = 0$

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\bot$ derived

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Can represent refutation as

- annotated list or
- directed acyclic graph

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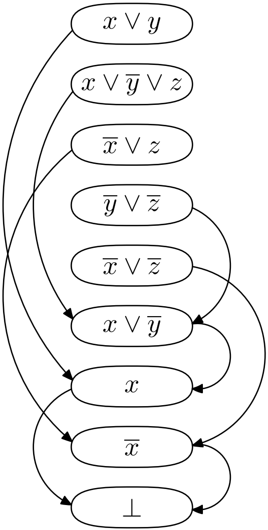

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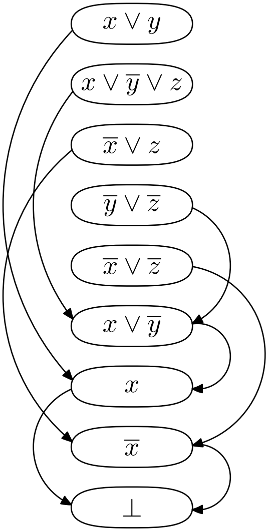
Tree-like resolution if DAG is tree



Resolution Size/Length and Width

Size/length $= \#$ clauses in refutation [9 in our example]

Most fundamental measure in proof complexity
Never worse than $\exp(\mathcal{O}(\#\text{variables}))$
Matching $\exp(\Omega(\text{formula size}))$ lower bounds known

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Never worse than $\exp(\mathcal{O}(\#\text{variables}))$
Matching $\exp(\Omega(\text{formula size}))$ lower bounds known

Width = size of largest clause in refutation [3 in our example]

Always $\leq$ #variables
Helpful measure to get a handle on size (as we shall soon see)

Polynomial Calculus (PC)

From [CEI96]; with adjustment in [ABRW02]

Clauses interpreted as polynomials over field $\mathbb{F}$
(Evaluate to true $\equiv$ vanish)

Example: $x \vee y \vee \overline{z}$ gets translated to $\overline{x}\overline{y}z$

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Derivation rules

Boolean axioms $\dfrac{}{x^2 - x}$

Negation $\dfrac{}{x + \overline{x} - 1}$

Linear combination $\dfrac{p \quad q}{\alpha p + \beta q}$

Multiplication $\dfrac{p}{xp}$

Goal: Derive $1 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

Formalizes Gröbner basis computations

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Clauses turn into monomials
Write out all polynomials as sums of monomials
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Size — analogue of resolution length/size
total # monomials in refutation counted with repetitions

Degree — analogue of resolution width
largest degree of monomial in refutation

Polynomial Calculus Stronger than Resolution

Polynomial calculus can simulate resolution proofs efficiently with respect to both size and width/degree

- Can mimic resolution refutation step by step
- Hence worst-case upper bounds for resolution carry over

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simulated by polynomial calculus derivation:

$$\cfrac{\overline{x}y\overline{z} \qquad \cfrac{\cfrac{yz}{\overline{x}yz} \qquad \cfrac{\cfrac{z + \overline{z} - 1}{yz + y\overline{z} - y}}{\overline{x}yz + \overline{x}y\overline{z} - \overline{x}y}}{-\overline{x}y\overline{z} + \overline{x}y}}{\overline{x}y}$$

Examples of Some Hard Formulas (1/3)

Random k-CNF formulas

Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

Exponential size lower bounds for for

- resolution [CS88, BKPS02]
- polynomial calculus over fields of characteristic $\neq 2$ [BI99]
- polynomial calculus over any field [AR03]

Examples of Some Hard Formulas (2/3)

Pigeonhole principle (PHP)

"$n + 1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
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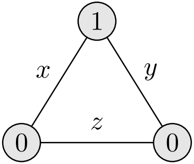
- All PHP versions exponentially hard for resolution [Hak85]
- "Vanilla PHP" exponentially hard for PC [AR03]
- Onto functional PHP easy for PC (over any field) [Rii93]
- What about functional PHP and onto PHP for PC?

Examples of Some Hard Formulas (3/3)

Tseitin formulas

"Sum of degrees of vertices in graph is even"

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



$$
\begin{aligned}
&(x \lor y) && \land (\overline{x} \lor z) \\
\land\ &(\overline{x} \lor \overline{y}) && \land (y \lor \overline{z}) \\
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\end{aligned}
$$

- Exponentially hard for resolution on expanders [Urq87]
- And for polynomial calculus in characteristic $\neq 2$ [BGIP01]
- But PC over $\mathrm{GF}(2)$ can do Gaussian elimination

Upper Bounds from Resolution Width and PC Degree

Width/degree upper bound ⇒ size upper bound

Resolution: At most $(2 \cdot \#\text{variables})^{\text{width}}$ distinct clauses

Polynomial calculus: Essentially same bound; more careful argument [CEI96]

These simple upper bounds are essentially tight [ALN16]

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Width/degree lower bound ⇒ size lower bound

Much less obvious. . .

Width/Degree Lower Bounds Imply Size Lower Bounds

Theorem ([IPS99, BW01])

For k-CNF formula over N variables

$$\textit{proof size} \geq \exp\left(\Omega\left(\frac{(\textit{proof width/degree})^2}{N}\right)\right)$$

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Resolution

- Well-developed machinery for width lower bounds
- One of many available tools

Polynomial calculus

- Degree lower bound machinery way less developed
- And pretty much only tool?!

Conversion to k-CNF "Graph Versions" of Formulas

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- **Solution:** Restrict formulas to bounded-degree graphs

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For graph (onto functional) PHP, pigeons can fly only to neighbour holes:

$\bigvee_{j \in \mathcal{N}(i)} p_{i,j}$	pigeon i goes into hole in $\mathcal{N}(i)$
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- Now strong width lower bounds $\Rightarrow$ strong size lower bounds
- And size lower bounds hold for original, unrestricted formulas
- Lower bounds for graph PHP also of independent interest

Lower Bounds via Graph Expansion

Standard approach:

Lower bounds from expansion

Simplest example is the clause-variable incidence graph (CVIG)

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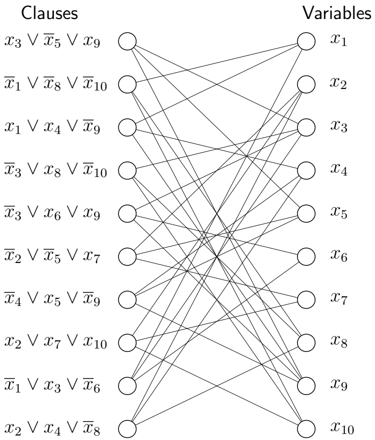

Lower Bounds via Graph Expansion

Standard approach:

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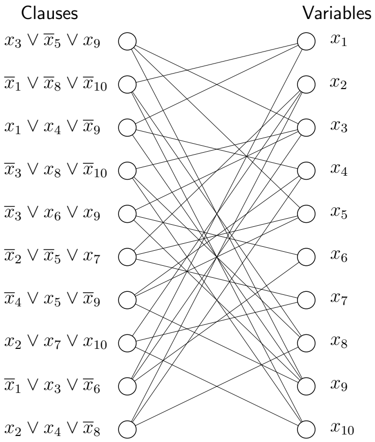
Boundary expansion:

Subsets of left vertices have many unique right neighbours



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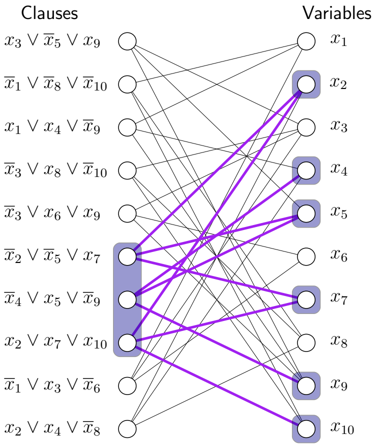

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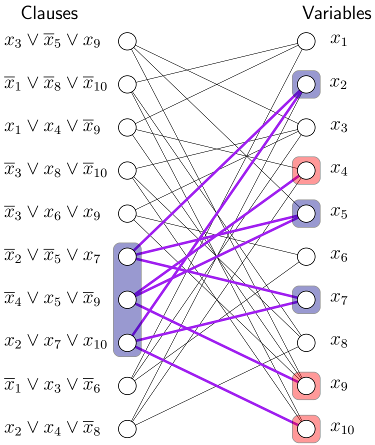

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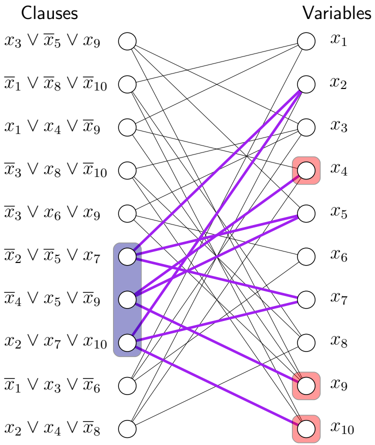

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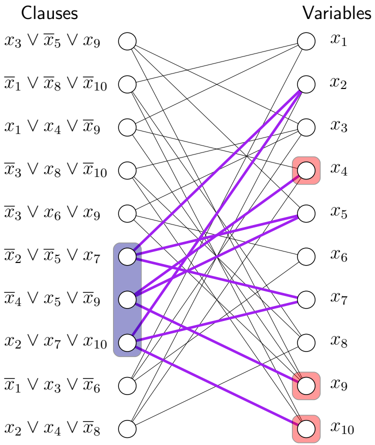

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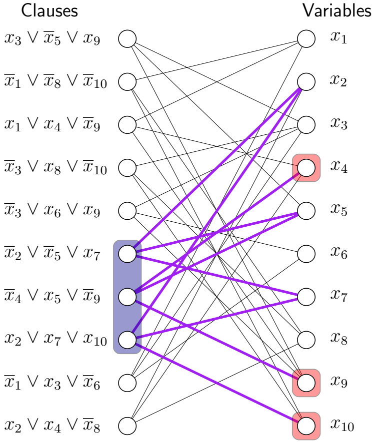
Need graph capturing combinatorial structure!



Generalized Incidence Graphs for CNF Formulas

Given CNF formula $\mathcal{F}$ over variables $\mathcal{V}$

- Partition clauses into $\mathcal{F} = E \cup \bigcup_{i=1}^{m} F_i$ (for E satisifiable)
- Divide variables into $\mathcal{V} = \bigcup_{j=1}^{n} V_j$ — **not** always partition
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Build bipartite $(\mathcal{U}, \mathcal{V})_E$-graph $\mathcal{G}$

- Left vertices $\mathcal{U} = \{F_1, \ldots, F_m\}$
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E not part of graph, but "filters" which assignments to consider
(E.g., partial matchings for pigeonhole principle formulas)

The Resolution Edge Game

Resolution edge game on (F_i, V_j) w.r.t. "filtering set" E

- Adversary choses any total assignment α such that $\alpha(E) = 1$
- We can modify α on V_j to get α'
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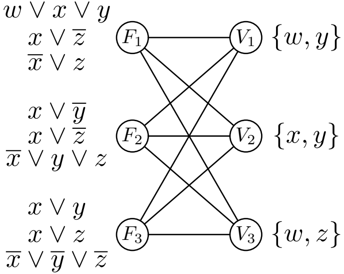

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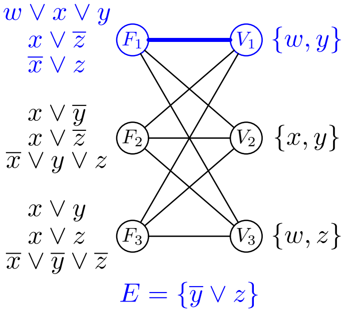

Edge game on (F_1, V_1) w.r.t. E

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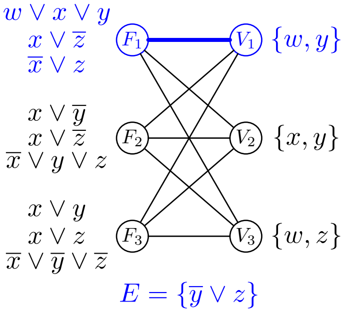

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Take $\alpha_1 = \{w = y = z = 0, x = 1\}$

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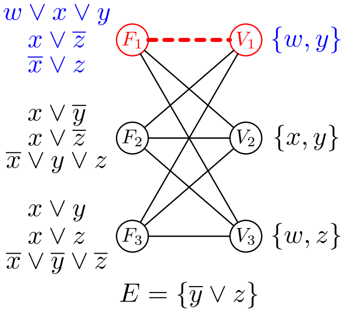

Edge game on (F_1, V_1) w.r.t. E

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Can't win, since

- $\alpha_1(\overline{x} \vee z) = 0$
- can't flip x or z (not in V_1)

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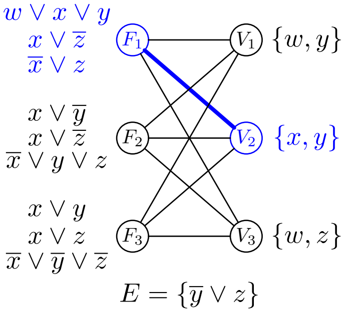

Edge game on (F_1, V_2) w.r.t. E

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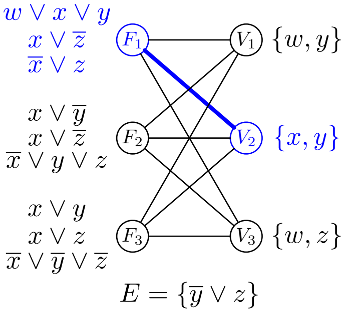

Edge game on (F_1, V_2) w.r.t. E

Take $\alpha_2 = \{w = y = z = 0, x = *\}$

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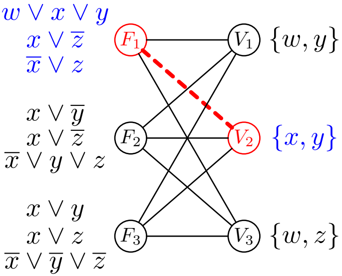

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Edge game on (F_1, V_2) w.r.t. E

Take $\alpha_2 = \{w = y = z = 0, x = *\}$

Again can't win, since

- can't flip w or z (not in V_2)
- flipping $y \in V_2$ falsifies E
- $F_1{\restriction_{\{w=y=z=0\}}} = \{x, \overline{x}\}$

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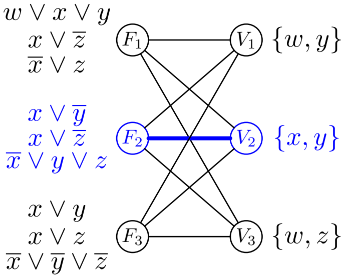

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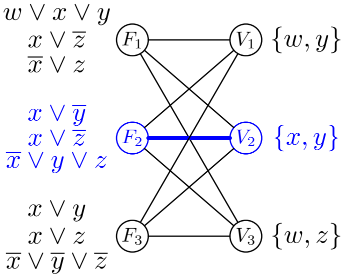

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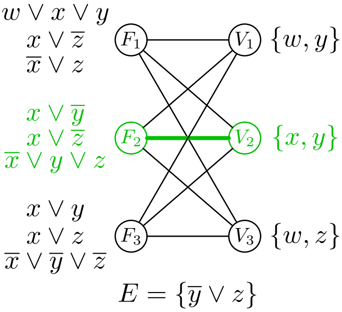

Edge game on (F_2, V_2) w.r.t. E

Now we can win!

Given any α_3 s.t. $\alpha_3(E) = 1$:

- assign $\alpha'(x) = \alpha_3(y \vee z)$
- E still OK — didn't touch y, z
- F_2 OK — encodes $x \leftrightarrow (y \vee z)$

Edge Game, Expansion, and Width Lower Bounds

Recall boundary $\partial(\mathcal{U}') = \{V \in \mathcal{N}(\mathcal{U}') \mid \mathcal{N}(V) \cap \mathcal{U}' = \{F\} \text{ unique}\}$

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Resolution expander

Say that an $(\mathcal{U}, \mathcal{V})_E$-graph is an (s, δ, E)-resolution expander if

- For all $\mathcal{U}' \subseteq \mathcal{U}$, $|\mathcal{U}'| \leq s$ it holds that $|\partial(\mathcal{U}')| \geq \delta|\mathcal{U}'|$
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Theorem (essentially [BW01])

If the CNF formula $\mathcal{F}$ admits an (s, δ, E)-resolution expander with overlap ℓ, then

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For every $C_i \in \pi$, define "support" $Sup_s(C_i) \subseteq \mathcal{F} \setminus E$ such that

1. $|Sup_s(C_i)| \leq s/2$
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$\Rightarrow$ $\left|Sup_s(C_i)\right|$ so small that $Sup_s(C_i) \cup E$ satisfiable

$\Rightarrow$ $Sup_s(C_i) \cup E \vDash C_i$ means that C_i satisfiable (hence not $\perp$) $\square$

Support

Clause neighbourhood $\mathcal{N}(C) = \{V \in \mathcal{V} \mid \mathit{Vars}(C) \cap V \neq \emptyset\}$

Left-side set $\mathcal{U}' \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathcal{V})_E$-graph is (s, C)-contained if

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s-support $\mathit{Sup}_s(C)$ of C = largest (s, C)-contained subset
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Need to argue:

- $Sup_s(C_i)$ well-defined — by expansion
- $|Sup_s(C_i)| \leq s/2$ — also by expansion
- $Sup_s(C_i) \cup E \vDash C_i$ — by resolution edge game and induction

Applications: Tseitin and Onto-FPHP

Tseitin formulas

- F_i = clauses encoding parity constraint for ith vertex
- V_j = singleton set with jth edge (so overlap $\ell = 1$)
- $E = \emptyset$
- If underlying graph edge expander, then $(\mathcal{U}, \mathcal{V})_E$-graph is resolution expander with same parameters

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Onto functional PHP formulas

- F_i = singleton set with pigeon axiom for pigeon i
- V_j = all variables $p_{i,j}$ mentioning hole j (again overlap $\ell = 1$)
- E = all hole, functional, and onto axioms
- If onto FPHP restricted to bipartite graph, then $(\mathcal{U}, \mathcal{V})_E$-graph is resolution expander with same parameters

From Resolution to Polynomial Calculus

So far: Obtain resolution width lower bounds from expander graphs where we can win following game on all edges

Resolution edge game on (F, V) with respect to E

1. Adversary provides total assignment α such that $\alpha(E) = 1$
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Polynomial calculus degree lower bounds require harder game

Polynomial calculus edge game on (F, V) with respect to E

1. Commit to partial assignment $\rho_V : V \to \{0, 1\}$
2. Adversary provides total assignment α such that $\alpha(E) = 1$
3. Substituting ρ_V for V should yield $\alpha[\rho_V/V](F \wedge E) = 1$

The Polynomial Calculus Edge Game

To win PC edge game on (F, V), need to find $\rho_V : V \to \{0, 1\}$ s.t.

- $\rho_V(F) = 1$
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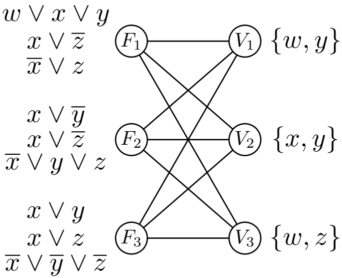

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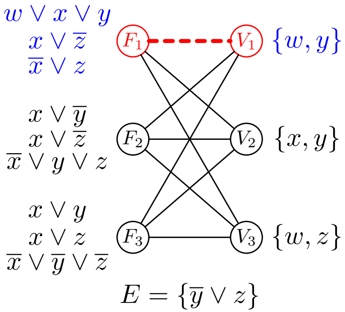

Recall that for resolution edge game we:

- Lose on (F_1, V_1)

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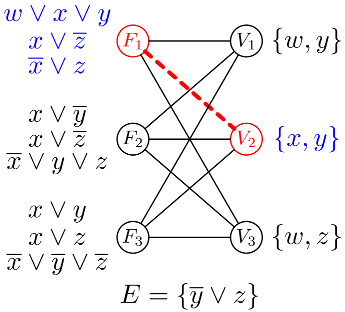

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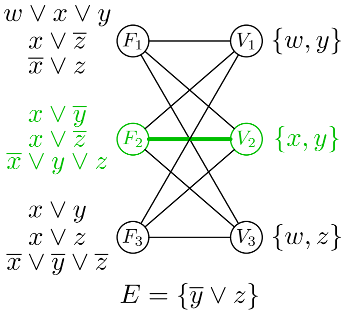

Recall that for resolution edge game we:

- Lose on (F_1, V_1)
- Lose on (F_1, V_2)
- Win on (F_2, V_2)

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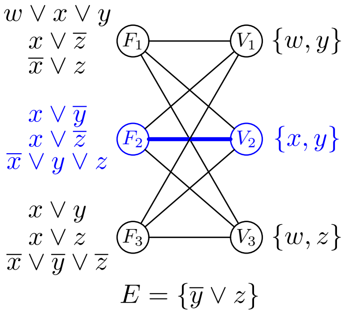

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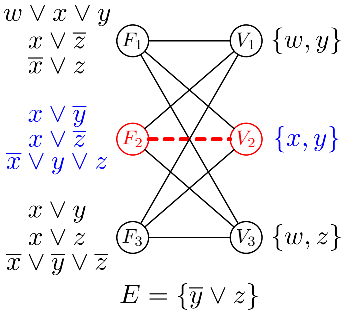

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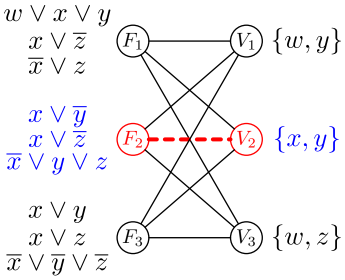

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PC edge game on (F_2, V_2) w.r.t. E

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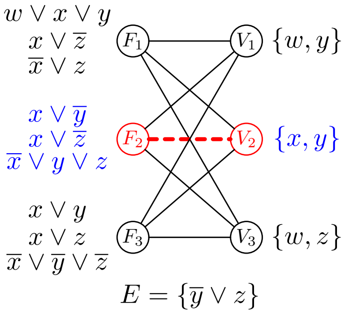

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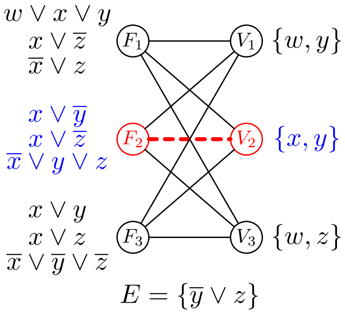

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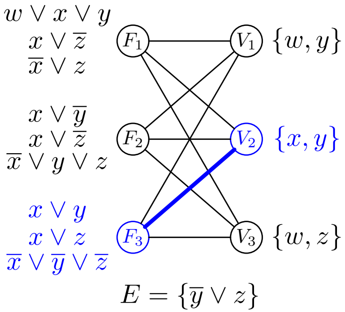

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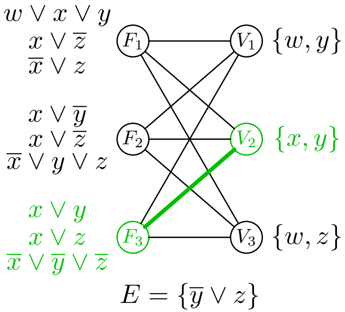

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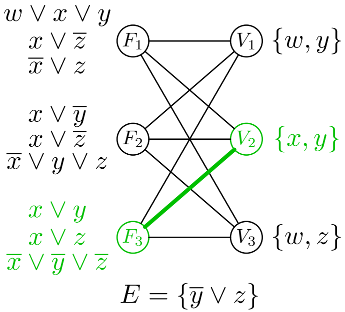

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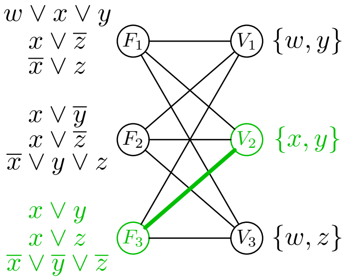

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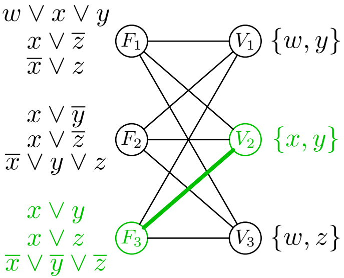

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A Generalized Method for PC Degree Lower Bounds

Polynomial calculus expander

Say that an $(\mathcal{U}, \mathcal{V})_E$-graph is an (s, δ, E)-PC expander if

- For all $\mathcal{U}' \subseteq \mathcal{U}$, $|\mathcal{U}'| \leq s$ it holds that $|\partial(\mathcal{U}')| \geq \delta|\mathcal{U}'|$
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If $\mathcal{F}$ admits an (s, δ, E)-PC expander with overlap ℓ, then

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Also holds for sets of polynomials not obtained from CNFs
Proof by carefully adapting [AR03] (fairly involved — can't say much)

Consequences

Common framework for previous lower bounds

- Random k-CNF formulas [AR03]
- CNF formulas with expanding CVIGs [AR03]
- "Vanilla" PHP formulas [AR03]
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New lower bounds

- Functional pigeonhole principle [MN15]
- Graph colouring [LN17]

Hardness of Different Flavours of PHP

Variant	Resolution	Polynomial calculus
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Joint work with Mladen Mikša [MN15]:

- Observe that [AR03] proves hardness of Onto-PHP
- Prove that functional PHP is hard for polynomial calculus (answering open question in [Raz02, Raz14])

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Theorem ([MN15])

If G is a (standard) bipartite (s, δ)-boundary expander with left degree $\leq d$, then $FPHP_G$ requires PC degree $> \delta s/(2d)$

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- So get same expansion parameters, and theorem follows □

Graph Colouring

Graph k-colouring formulas

"$G = (V, E)$ is k-colourable"

Variables $x_{v,c}$ = "vertex v gets colour c"

$x_{v,1} \lor x_{v,2} \lor \cdots \lor x_{v,k}$	every vertex v gets a colour
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On the contrary, [DLMM08, DLMO09, DLMM11, DMP$^+$15] claim very efficient algorithms based on Nullstellensatz ("static PC") for slightly different encoding using primitive kth roots of unity

Polynomial Calculus Lower Bound for Colouring

Joint work with Massimo Lauria [LN17]:

Theorem ([LN17])

For any $k \geq 3$ $\exists$ constant-degree graphs which require linear PC degree, and hence exponential size, to be proven non-k-colourable

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Lower bound applies also to kth-root-of-unity encoding

Answers open question raised in [DLMO09, LLO16]

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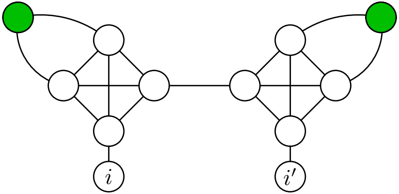

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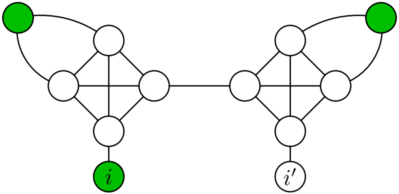

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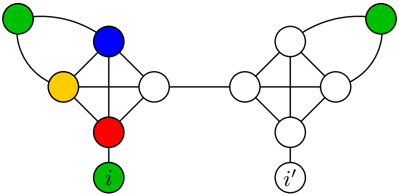

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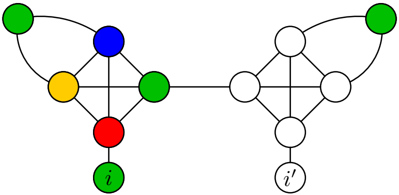

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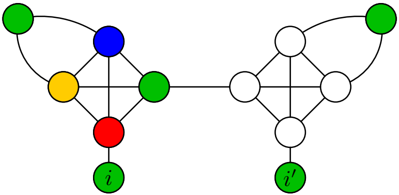

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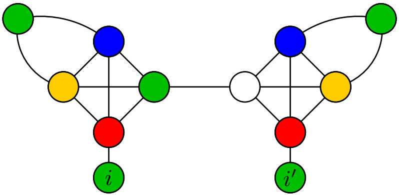

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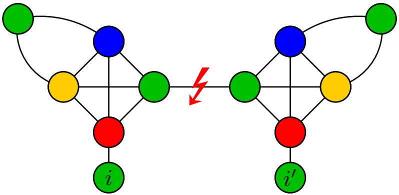

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Symmetric argument in right subgadget $\Rightarrow$ **contradiction**

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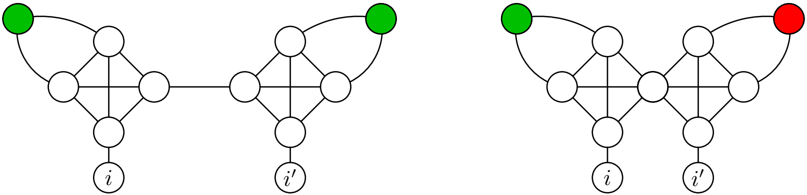

not i and i' both green

not i green and i' red

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- Find truly general framework capturing all PC degree bounds
  - We generalize only part of [AR03]
  - Cannot handle characteristic-dependent bounds à la [BGIP01]
  - Combination of [AR03] and [MN15] might give lower bounds for even colouring formulas [Mar06, VEG$^+$18]

Take-away Message

Generalized method for width and degree lower bounds

- Unified framework for most previous lower bounds
- Highlights similarities and differences between resolution and polynomial calculus
- Exponential polynomial calculus size lower bound for
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Future directions

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Thank you for your attention!

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