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Recent Progress on Representation Theoretic Multiplicities in GCT

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1. Weakness of occurrence obstructions (with Bürgisser and Panova)
2. Multiplicities are strictly stronger than occurrences (with Dörfler and Panova)
3. Using multiplicities: connecting orbits with their closures (with Kandasamy)

Orbit closures of determinant and permanent

- $\det_n := \sum_{\pi\in\mathfrak{S}_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}$, $\operatorname{per}_m := \sum_{\pi\in\mathfrak{S}_m} \prod_{i=1}^{m} x_{i,\pi(i)}$
- For a linear map $g : \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$ define $g\det_n$ via $(g\det_n)(x) := \det_n(g^t(x))$
- $\mathbb{C}^{n^2\times n^2}\det_n = \{$determinants of $n \times n$ matrices whose entries are homog. lin. polyn.$\}$

Example:

$$\det\begin{pmatrix} x_{1,1}+x_{1,2} & x_{1,2}-2x_{2,2} \\ x_{2,1} & x_{1,1}+x_{1,2} \end{pmatrix} = x_{1,1}^2 + 2x_{1,1}x_{1,2} + x_{1,2}^2 - x_{1,2}x_{2,1} + 2x_{2,1}x_{2,2} \in \mathbb{C}^{4\times 4}\det_2$$

- Valiant 1979: For all m there exists $n \geq m$ such that $x_{1,1}^{n-m}\operatorname{per}_m \in \mathbb{C}^{n^2\times n^2}\det_n$.
- Closure: $\overline{\mathbb{C}^{n^2\times n^2}\det_n} = \overline{\mathsf{GL}_{n^2}\det_n}$
- Define $\underline{\mathrm{dc}}(\operatorname{per}_m)$ to be the smallest n such that $x_{1,1}^{n-m}\operatorname{per}_m \in \overline{\mathsf{GL}_{n^2}\det_n}$.
- GCT Conjecture: $\underline{\mathrm{dc}}(\operatorname{per}_m)$ grows superpolynomially.

Observation:

$$x_{1,1}^{n-m}\operatorname{per}_m \in \overline{\mathsf{GL}_{n^2}\det_n} \quad \text{iff} \quad \overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\operatorname{per}_m)} \subseteq \overline{\mathsf{GL}_{n^2}\det_n}.$$

Example of a (weak) lower bound technique:
If $\dim \overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\operatorname{per}_m)} > \dim \overline{\mathsf{GL}_{n^2}(\det_n)}$, then $\underline{\mathrm{dc}}(\operatorname{per}_m) > n$.

Coordinate rings

- $\mathrm{Poly}^n\mathbb{C}^N :=$ homog. degree n polyn. in N variables.
- $\dim \mathrm{Poly}^n\mathbb{C}^N = \binom{N+n-1}{n}$
- $\mathbb{C}[\mathrm{Poly}^n\mathbb{C}^N]_d :=$ homog. degree d polyn. in $\binom{N+n-1}{n}$ many variables
- Example: $n = N = 2$
  - $\mathrm{Poly}^2\mathbb{C}^2$ has basis $\{x^2, xy, y^2\}$.
  - Every element in $\mathrm{Poly}^2\mathbb{C}^2$ can be expressed as $ax^2 + bxy + cy^2$
  - $\mathbb{C}[\mathrm{Poly}^2\mathbb{C}^2]_2$ has basis $\{a^2, ab, ac, b^2, bc, c^2\}$
  - The discriminant $b^2 - 4ac \in \mathbb{C}[\mathrm{Poly}^2\mathbb{C}^2]_2$
  - $b^2 - 4ac = 0$ iff $ax^2 + bxy + cy^2 = (\alpha x + \beta y)^2$ for some $\alpha, \beta \in \mathbb{C}$
- Action of GL_N on $\mathbb{C}[\mathrm{Poly}^n\mathbb{C}^N]_d$: Define $(gf)(p) := f(g^t p)$

For $Z \subseteq \mathrm{Poly}^n\mathbb{C}^N$, define the coordinate ring:

$$\mathbb{C}[\overline{Z}] := \mathbb{C}[\mathrm{Poly}^n\mathbb{C}^N]_{|_{\overline{Z}}} \qquad \text{(restrict domain of definition to } \overline{Z}\text{)}$$

If $\overline{Y} \subseteq \overline{Z}$, then this gives a natural surjection:

$$\mathbb{C}[\overline{Z}] \twoheadrightarrow \mathbb{C}[\overline{Y}]$$

If $\overline{Z}$ is closed under the action of GL_N, then $\mathbb{C}[\overline{Z}]$ inherits the action of GL_N.

Obstructions based on representation theoretic multiplicities

- Goal: To prove $\overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)} \not\subseteq \overline{\mathsf{GL}_{n^2}\det_n}$
- If $\overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)} \subseteq \overline{\mathsf{GL}_{n^2}\det_n}$, then $\mathbb{C}[\overline{\mathsf{GL}_{n^2}\det_n}]_d \twoheadrightarrow \mathbb{C}[\overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)}]_d$

The group action of GL_{n^2} lets us decompose into irreducibles:

- $\mathbb{C}[\overline{\mathsf{GL}_{n^2}\det_n}]_d \quad = \quad \bigoplus_\lambda \mathscr{V}_\lambda^{\oplus \mathsf{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_{n^2}\det_n}]_d)}$,
- $\mathbb{C}[\overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)}]_d = \bigoplus_\lambda \mathscr{V}_\lambda^{\oplus \mathsf{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)}]_d)}$

Since the surjection is GL_{n^2}-equivariant, Schur's lemma implies:

$$\mathsf{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_{n^2}\det_n}]_d) \geq \mathsf{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)}]_d).$$

Multiplicity obstruction:
If $\exists\lambda$ with $\mathsf{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_{n^2}\det_n}]_d) < \mathsf{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)}]_d)$, then $\underline{\mathrm{dc}}(\mathrm{per}_m) > n$.
Occurrence obstruction:
If $\exists\lambda$ with $\mathsf{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_{n^2}\det_n}]_d) = 0 < \mathsf{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)}]_d)$, then $\underline{\mathrm{dc}}(\mathrm{per}_m) > n$.

Theorem [Bürgisser, I, Panova 2016], disproving a conj. by Mulmuley and Sohoni

There are no occurrence obstrucions that prove $\underline{\mathrm{dc}}(\mathrm{per}_m) \geq m^{25}$.

Proof relies on the padding of the permanent.
Replace $\det_n$ by homogeneous iterated matrix multiplication to avoid this: Boot camp talk

Summary of part 1

- If $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{n^2}\det_n}]_d) < \mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)}]_d)$, then $\underline{\mathrm{dc}}(\mathrm{per}_m) > n$.
- Occurrence obstruction: $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{n^2}\det_n}]_d) = 0 < \mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{n^2}(x_{1,1}^{n-m}\mathrm{per}_m)}]_d)$
- But $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{n^2}\det_n}]_d) > 0$ in all relevant cases, so that $\underline{\mathrm{dc}}(\mathrm{per}_m) > m^{25}$ cannot be proved using occurrence obstructions.
- The proof works in all computational models that involve padding.

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Good news: There are group varieties that

- cannot be separated with occurrence obstructions, but
- can be separated with multiplicity obstructions.

(no padding involved)

Factorizing power sums

Two GL_m-varieties:

- Product of homogeneous linear forms:
$$\mathsf{Ch}_m^n := \{\ell_1 \cdots \ell_n \mid \ell_i \in \mathrm{Poly}^1\mathbb{C}^m\} \subseteq \mathrm{Poly}^n\mathbb{C}^m.$$
- Border Waring rank $\leq k$ polynomials:
$$\mathsf{Pow}_{m,k}^n := \overline{\{\ell_1^n + \cdots + \ell_k^n \mid \ell_i \in \mathrm{Poly}^1\mathbb{C}^m\}} \subseteq \mathrm{Poly}^n\mathbb{C}^m.$$

Theorem [Dörfler, I, Panova 2019]

For any $m \geq 3$, $n \geq 2$, let $k = d = n+1$, $\lambda = (n^2-2, n, 2)$. Then
$$\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_d) < \mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Pow}_{m,k}^n]_d),$$
i.e., λ is a multiplicity obstruction that shows $\mathsf{Pow}_{m,k}^n \not\subseteq \mathsf{Ch}_m^n$.

In a finite case we can rule out the existence of occurrence obstructions:

Theorem [Dörfler, I, Panova 2019]

Let $k = 4$, $n = 6$, $m = 3$, $d = 7$, $\lambda = (n^2-2, n, 2) = (34, 6, 2)$. Then
$$\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_d) = 7 < 8 = \mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Pow}_{m,k}^n]_d)$$
and hence
$$\mathsf{Pow}_{m,k}^n \not\subseteq \mathsf{Ch}_m^n.$$
For all μ: If $\mathrm{mult}_\mu(\mathbb{C}[\mathsf{Pow}_{m,k}^n]_d) > 0$, then $\mathrm{mult}_\mu(\mathbb{C}[\mathsf{Ch}_m^n]_d) > 0$.

No occurrence obstructions

- Goal: If $\mathrm{mult}_\mu(\mathbb{C}[\mathrm{Poly}^6\mathbb{C}^3]) > 0$, then $\mathrm{mult}_\mu(\mathbb{C}[\mathrm{Ch}_3^6]) > 0$.
- Partitions: $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{N}^3, \quad \mu_1 \geq \mu_2 \geq \mu_3$

Proposition (Semigroup properties)

Let μ and ν be partitions with $\mathrm{mult}_\mu(\mathbb{C}[\mathrm{Poly}^6\mathbb{C}^3]) > 0$ and $\mathrm{mult}_\nu(\mathbb{C}[\mathrm{Poly}^6\mathbb{C}^3]) > 0$. Then $\mathrm{mult}_{\mu+\nu}(\mathbb{C}[\mathrm{Poly}^6\mathbb{C}^3]) > 0$.

Let μ and ν be partitions with $\mathrm{mult}_\mu(\mathbb{C}[\mathrm{Ch}_3^6]) > 0$ and $\mathrm{mult}_\nu(\mathbb{C}[\mathrm{Ch}_3^6]) > 0$. Then $\mathrm{mult}_{\mu+\nu}(\mathbb{C}[\mathrm{Ch}_3^6]) > 0$.

Conclusion: $\{\mu \mid \mathrm{mult}_\mu(\mathbb{C}[\mathrm{Poly}^6\mathbb{C}^3]) > 0\}$ and $\{\mu \mid \mathrm{mult}_\mu(\mathbb{C}[\mathrm{Ch}_3^6]) > 0\}$ are semigroups.

$\{\mu \mid \mathrm{mult}_\mu(\mathbb{C}[\mathrm{Poly}^6\mathbb{C}^3]) > 0\}$ has 89 generators:

(6), (6, 6), (8, 4), (10, 2), (6, 6, 6), (8, 6, 4), (10, 4, 4), (9, 6, 3), (8, 8, 2), (10, 6, 2), (11, 5, 2), (10, 7, 1), (12, 4, 2), (11, 6, 1), (10, 8), (14, 2, 2), (13, 4, 1), (13, 5), (15, 3), (8, 8, 8), (10, 8, 6), (11, 7, 6), (10, 9, 5), (11, 8, 5), (10, 10, 4), (12, 7, 5), (11, 9, 4), (13, 6, 5), (12, 8, 4), (11, 10, 3), (13, 7, 4), (12, 9, 3), (13, 8, 3), (12, 10, 2), (15, 5, 4), (14, 7, 3), (13, 9, 2), (13, 10, 1), (16, 5, 3), (15, 7, 2), (14, 9, 1), (17, 4, 3), (15, 8, 1), (15, 9), (19, 3, 2), (18, 5, 1), (17, 7), (10, 10, 10), (11, 10, 9), (12, 10, 8), (13, 9, 8), (12, 11, 7), (13, 10, 7), (14, 9, 7), (13, 11, 6), (15, 8, 7), (13, 12, 5), (16, 7, 7), (15, 9, 6), (14, 11, 5), (13, 13, 4), (15, 10, 5), (15, 11, 4), (14, 13, 3), (16, 11, 3), (15, 13, 2), (15, 14, 1), (17, 13), (13, 12, 11), (14, 11, 11), (13, 13, 10), (15, 11, 10), (14, 13, 9), (16, 11, 9), (15, 13, 8), (15, 14, 7), (18, 9, 9), (15, 15, 6), (17, 17, 2), (18, 17, 1), (26, 5, 5), (15, 14, 13), (16, 13, 13), (15, 15, 12), (17, 17, 8), (18, 15, 15), (17, 17, 14), (25, 23), (45, 45).

For each generator μ we construct an occurrence of $\mathscr{V}_\mu$ in $\mathbb{C}[\mathrm{Ch}_3^6]$ by computer.

Multiplicity obstructions exist

Theorem [Dörfler, I, Panova 2019]

For any $m \geq 3$, $n \geq 2$, let $k = d = n + 1$, $\lambda = (n^2 - 2, n, 2)$. Then

$$\text{mult}_\lambda(\mathbb{C}[\text{Ch}_m^n]_d) < \text{mult}_\lambda(\mathbb{C}[\text{Pow}_{m,k}^n]_d),$$

i.e., λ is a multiplicity obstruction that shows $\text{Pow}_{m,k}^n \not\subseteq \text{Ch}_m^n$.

Proof:

The **plethysm coefficient** $a_\lambda(d, n) := \text{mult}_\lambda(\mathbb{C}[\text{Poly}^n\mathbb{C}^N]_d)$

Proposition [Bürgisser, I, Panova 2016]

If $k \geq d$, then $\text{mult}_\lambda(\mathbb{C}[\text{Pow}_{m,k}^n]_d) = a_\lambda(d, n)$.

In other words: $\text{Pow}_{m,k}^n$ is a hitting set for degree $\leq k$ polynomials

Remains to show: $\text{mult}_\lambda(\mathbb{C}[\text{Ch}_m^n]_d) < a_\lambda(d, n)$

Remains to show: $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ch}^n_m]_d) < a_\lambda(d,n)$ for $d = n+1$, $\lambda = (n^2-2, n, 2)$

Use inheritance theorem: $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ch}^n_m]) = \mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_n(x_1\cdots x_n)}])$

$\mathbb{C}[\mathrm{GL}_n(x_1\cdots x_n)]$:= rational functions that are defined everywhere on $\mathrm{GL}_n(x_1\cdots x_n)$.
$\mathbb{C}[\overline{\mathrm{GL}_n(x_1\cdots x_n)}] \subseteq \mathbb{C}[\mathrm{GL}_n(x_1\cdots x_n)]$, in particular

$$\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_n(x_1\cdots x_n)}]) \leq \mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_n(x_1\cdots x_n)]).$$

$$\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_n(x_1\cdots x_n)]) = \underbrace{\dim \mathscr{V}_{\lambda^*}^H}_{=a_\lambda(n,d)} \qquad \text{for } |\lambda| = nd,$$

where $H \subseteq \mathrm{GL}_n$ is the stabilizer of $x_1\cdots x_n$.

Proof:

$$\mathbb{C}[\mathrm{GL}_n(x_1\cdots x_n)] = \mathbb{C}[\mathrm{GL}_n/H] = \mathbb{C}[\mathrm{GL}_n]^H \overset{\substack{\text{Algebraic}\\ \text{Peter-Weyl}}}{=} \bigoplus_\lambda \mathscr{V}_\lambda \otimes \mathscr{V}_{\lambda^*}^H \qquad \square$$

Proposition (proof based on symmetric functions)

For $\lambda = (n^2-2, n, 2)$: $a_\lambda(n+1, n) = 1 + a_\lambda(n, n+1)$.

$$\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ch}^n_m]) = \mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_n(x_1\cdots x_n)}]) \leq \mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_n(x_1\cdots x_n)]) = a_\lambda(n,d) < a_\lambda(d,n).\square$$

Summary of part 2

- $\text{mult}_\lambda(\mathbb{C}[\text{Ch}^n_m]_d) < \text{mult}_\lambda(\mathbb{C}[\text{Pow}^n_{m,k}]_d)$, therefore $\text{Pow}^n_{m,k} \not\subseteq \text{Ch}^n_m$.
- Proof based on relationship "orbit vs orbit closure":
  $\text{mult}_\lambda(\mathbb{C}[\overline{\text{GL}_n(x_1 \cdots x_n)}]) \leq \text{mult}_\lambda(\mathbb{C}[\text{GL}_n(x_1 \cdots x_n)])$.
- In a finite case we verified by computer:
  there are no occurrence obstructions showing $\text{Pow}^n_{m,k} \not\subseteq \text{Ch}^n_m$, but multiplicity obstructions work

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[Bürgisser, I 2017] connects orbit and orbit closure more closely:

- Let $0 \neq \Phi \in \mathbb{C}[\overline{\mathrm{GL}_m(x_1^n + \cdots + x_m^n)}]$ be invariant under SL_m ($x_1^n + \cdots + x_m^n$ is not in the null cone)
- Then Φ is nonzero everywhere on $\mathrm{GL}_m(x_1^n + \cdots + x_m^n)$
- It turns out: Φ vanishes on the boundary $\overline{\mathrm{GL}_m(x_1^n + \cdots + x_m^n)} \setminus \mathrm{GL}_m(x_1^n + \cdots + x_m^n)$ (proof uses Hilbert-Mumford criterion and refinement by Luna and Kempf)
- As a result:
$$\mathbb{C}[\mathrm{GL}_m(x_1^n + \cdots + x_m^n)] = \mathbb{C}[\overline{\mathrm{GL}_m(x_1^n + \cdots + x_m^n)}]_\Phi$$
is the localization at Φ.

Theorem [Bürgisser, I 2017]

For all d there is e:

$$\mathbb{C}[\mathrm{GL}_m(x_1^n+\cdots+x_m^n)]_d \overset{\gamma}{\hookrightarrow} \mathbb{C}[\overline{\mathrm{GL}_m(x_1^n + \cdots + x_m^n)}]_{d+em} \subseteq \mathbb{C}[\mathrm{GL}_m(x_1^n+\cdots+x_m^n)]_{d+em},$$

where $\gamma(f) := \Phi^e f$.

Theorem [I, Kandasamy 2019]

For even n, an upper bound on the required e is $m + 4\frac{d}{n}$.

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Theorem [I, Kandasamy 2019]

For even n, an upper bound on the required e is $m + 4\frac{d}{n}$.

- Given a Young tableau T, we can explicitly construct a function f_T in $\mathbb{C}[\overline{\mathrm{GL}_m(x_1^n + \cdots + x_m^n)}]$
- All highest weight functions in $\mathbb{C}[\overline{\mathrm{GL}_m(x_1^n + \cdots + x_m^n)}]$ can be constructed in this way
- We have a combinatorial/linear algebra way of evaluating at points

We have a similar situation in $\mathbb{C}[\mathrm{GL}_m(x_1^n + \cdots + x_m^n)]$:

- Given a Young tableau S, we can explicitly construct a function f_S in $\mathbb{C}[\mathrm{GL}_m(x_1^n + \cdots + x_m^n)] \simeq \mathscr{V}_\lambda^H$
- All highest weight functions in $\mathbb{C}[\mathrm{GL}_m(x_1^n + \cdots + x_m^n)]$ can be constructed in this way
- We have a combinatorial/linear algebra way of evaluating at points

- Proof idea of I-Kandasamy: Given a tableau S, construct a slightly larger tableau T such that f_T and f_S coincide on $\mathrm{SL}_m(x_1^n + \cdots + x_m^n)$.

Summary of part 3

- The representation theory of $\mathbb{C}[\mathrm{GL}_N p]$ can usually be much better understood than the representation theory of $\mathbb{C}[\overline{\mathrm{GL}_N p}]$
- In many cases of interest: the representation theory of $\mathbb{C}[\mathrm{GL}_N p]$ and $\mathbb{C}[\overline{\mathrm{GL}_N p}]$ is connected by a fundamental invariant Φ
- In the case of power sums, this connection is very close
- The hope is that $\mathbb{C}[\mathrm{GL}_N p]$ and $\mathbb{C}[\overline{\mathrm{GL}_N p}]$ are closely related in more involved cases

Where does the hope for multiplicities come from?

Let $H \subseteq \mathrm{GL}_N$ be the stabilizer of p.

$$\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_N p]) = \dim \mathscr{V}_\lambda^H$$

Theorem [Larsen and Pink 1990, Inventiones math.]

$H \subseteq \mathrm{GL}_N$. Under reasonable assumptions, the group H is determined (up to group isomorphism) by the dimensions $\dim \mathscr{V}_\lambda^H$.

Pick H to be the stabilizer of a point p that is **characterized by its stabilizer**:

- determinant
- permanent
- iterated matrix multiplication polynomial
- power sum polynomial
- multilinear monomial
- matrix multiplication tensor
- unit tensor

Conclusion: A strengthening of this theorem would yield that p is characterized by its multiplicities.

Summary

- In the computational models with padding there are no occurence obstructions that prove strong lower bounds
- The padding can be removed: Iterated matrix multiplication
- But even in small explicit unpadded cases: multiplicity obstructions are stronger than occurrence obstructions
- Multiplicities in $\mathbb{C}[\mathrm{GL}_N p]$ can be studied with algebraic combinatorics.
  The connection to $\mathbb{C}[\overline{\mathrm{GL}_N p}]$ is hopefully close.
  (This works for power sums)
- Larsen and Pink: Give hope for multiplicity obstructions

Thank you for your attention!