An instance of symbolic determinant identity testing via $*$-algebras

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4 Dec 2018, Algebraic Methods Workshop @ Simons Institute

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ϵ-symmetrizable matrix tuples

- $\mathbb{F}$ is of characteristic $\neq 2$ and large enough.
- $\mathrm{M}_n(\mathbb{F})$: the linear space of $n \times n$ matrices.
- A *matrix space* is a linear subspace of $\mathrm{M}_n(\mathbb{F})$.

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Definition

$\vec{A} = (A_1, \ldots, A_m) \in \mathrm{M}_n(\mathbb{F})^m$ is *ϵ-symmetrizable*, if $\exists C, D \in \mathrm{GL}_n(\mathbb{F})$, such that every CA_iD is ϵ-symmetric.

The ϵ-symmetrization problem and polynomial identity testing

Recall: given $\vec{A} = (A_1, \ldots, A_m) \in \mathrm{M}_n(\mathbb{F})^m$, decide whether $\exists C, D \in \mathrm{GL}_n(\mathbb{F})$, such that every CA_iD is ϵ-symmetric.

1. Enough to search for $E \in \mathrm{GL}_n(\mathbb{F})$, such that every EA_i is ϵ-symmetric.
   - As $D^{-t}CA_i = D^{-t}(CA_iD)D^{-1}$ is also ϵ-symmetric.
2. Let $L(\vec{A}) := \{E \in \mathrm{M}(n, \mathbb{F}) : EA_i = \epsilon A_i^t E^t\}$. Then $L(\vec{A})$ is a matrix space.
3. The problem reduces to decide whether $L(\vec{A})$ contains a full-rank matrix. This is an instance of the symbolic determinant identity testing (SDIT) problem.
   - As $\mathbb{F}$ is large enough, this problem admits a randomized efficient algorithm.

Main result

Theorem

There exists a deterministic efficient algorithm that:

- *Given $n \times n$ matrices $A_1, \ldots, A_m$;*
- *Decide whether there exist invertible matrices C, D, such that every CA_iD is ϵ-symmetric.*

- Inspired by the $*$-algebra technique [Wilson'09] and the module isomoprhism techniques [Chistov-Ivanyos-Karpinski'97, Brooksbank-Luks'08, Ivanyos-Karpinski-Saxena'10].
- Our original motivation was from understanding singularity witnesses for matrix spaces beyond shrunk subspaces.

Motivation: singularity witnesses for singular matrix spaces

Singularity witnesses for singular matrix spaces

- By Kabanets-Impagliazzo, putting SDIT in NP already implies strong arithmetic circuit lower bounds.
- This amounts to finding small witnesses responsible for the singularity of a singularity matrix space.

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The non-commutative rank problem is concerned about one type of singularity witnesses, namely shrunk subspaces.

- $U \leq \mathbb{F}^n$ is a *shrunk subspace* for $\mathcal{A} \leq \mathrm{M}_n(\mathbb{F})^m$, if $\dim(\mathcal{A}(U)) < \dim(U)$, where $\mathcal{A}(U) = \langle \cup_{A \in \mathcal{A}} A(U) \rangle$.

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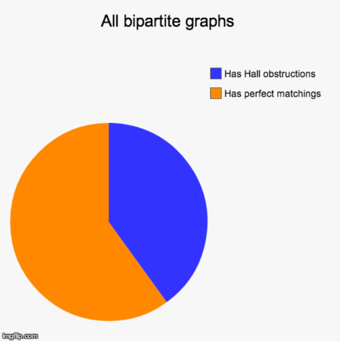
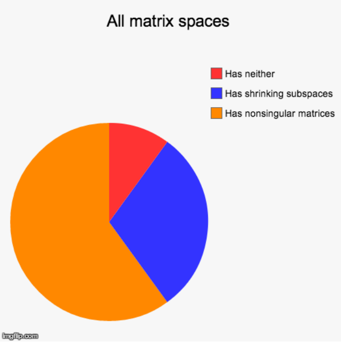
Matrix tuples with shrunk subspaces are the points in the nullcone of the left-right action by $\mathrm{SL}_n(\mathbb{F}) \times \mathrm{SL}_n(\mathbb{F})$ on $\mathrm{M}_n(\mathbb{F})^m$ [King,BD].

- Mulmuley conjectured that this problem could be put in P in GCT 5. Now it admits deterministic polynomial-time algorithms by [GGOW, IQS].

Singularity witnesses: shrunk subspaces are not enough

There are singular matrix spaces without shrunk subspaces: consider the space of 3×3 skew-symmetric matrices. That is, the analogue of Hall's marriage theorem does not hold.





Singularity witnesses beyond shrunk subspaces

Two classical examples from [Eisenbud-Harris, Lovász, Atkinson]:

1. Subspaces of the space of odd-size skew-symmetric matrices.
2. Skew-symmetric induced matrix spaces.
   - Given $n \times n$ skew-symmetric matrices $A_1, \ldots, A_n$, for $i \in [n]$, construct $B_i = [A_1 e_i, \ldots, A_n e_i]$, e_i the ith standard basis vector.
   - Then $\mathcal{B} = \langle B_1, \ldots, B_n \rangle$ is singular: $B = \alpha_1 B_1 + \cdots + \alpha_n B_n$ has $(\alpha_1, \ldots, \alpha_n)$ in the left kernel.

...and those spaces equivalent to them.

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Corollary

Given $\mathcal{B} = \langle B_1, \ldots, B_m \rangle \leq \mathrm{M}_n(\mathbb{F})$, there exists a deterministic efficient algorithm that decides whether $\mathcal{B}$ is equivalent to either a subspace of a skew-symmetric matrix space, or a skew-symmetric induced matrix space.

Tackling the ϵ-symmetrization problem

The strategy

Given $\vec{A} = (A_1, \ldots, A_m) \in \mathrm{M}_n(\mathbb{F})^m$, decide whether there is a full-rank matrix in $L^{\epsilon}(\vec{A}) = \{D \in \mathrm{M}_n(\mathbb{F}) : \forall i, D^t A_i = \epsilon A_i^t D\}$.

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- either conclude that D is of maximal rank;
- or find another $D' \in L^\epsilon(\vec{A})$ of higher rank.

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One simple rank increasing setting is the following.

- If $C, D \in \mathrm{M}_\ell(\mathbb{F})$, $C(\ker(D)) \not\subseteq \mathrm{im}(D)$.
- Then $\mathrm{rk}(C + \lambda D) > \mathrm{rk}(D)$ for all but at most ℓ $\lambda \in \mathbb{F}$.

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Essentially, we will show that, if D is not of maximal rank, then any linear basis of $L^\epsilon(\vec{A})$ contains a matrix that can be used as C.
...But not in the usual action!

The adjoint algebra of an ϵ-symmetric matrix tuple

Let $\vec{A} = (A_1, \ldots, A_m) \in S_n^\epsilon(\mathbb{F})^m$. We assume that $\vec{A}$ is *non-degenerate*, e.g. the common kernel of A_i's is trivial, and the union of images of A_i's spans the full space.

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Definition

Let $\vec{A} = (A_1, \dots, A_m) \in S_n^\epsilon(\mathbb{F})^m$. The *adjoint algebra* of $\vec{A}$ is

$$\mathrm{Adj}(\vec{A}) = \{D \in \mathrm{M}_n(\mathbb{F}) : \exists ! C \in \mathrm{M}_n(\mathbb{F}), \forall i, C^t A_i = A_i D\} \subseteq \mathrm{M}_n(\mathbb{F}).$$

$\mathrm{Adj}(\vec{A})$ admits an anti-automorphism $*$ of order 2, i.e. $D^* = C$.

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Algebras with anti-automorphisms of order 2 are termed as *involutive algebras* or $*$*-algebras*.

- Consider the transpose on $\mathrm{M}_n(\mathbb{F})$.

The $*$-symmetric elements of the adjoint algebra

Recall that for $\vec{A} \in S_n^\epsilon(\mathbb{F})^m$, we defined the adjoint algebra $\mathrm{Adj}(\vec{A})$.

Definition

The linear space of *$*$-symmetric elements* in $\mathrm{Adj}(\vec{A})$ is

$$\begin{aligned} \mathrm{Sym}^*(\vec{A}) &= \{D \in \mathrm{Adj}(\vec{A}) : D^* = D\} \\ &= \{D \in \mathrm{M}_n(\mathbb{F}) : \forall i, D^t A_i = A_i D\}. \end{aligned}$$

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Recall that for $\vec{A} \in \mathrm{M}_n(\mathbb{F})^m$, we defined

$$L^{\epsilon}(\vec{A}) = \{D \in \mathrm{M}_n(\mathbb{F}) : \forall i, D^t A_i = \epsilon A_i^t D\}.$$

So for $\vec{A} \in S_n^{\epsilon}(\mathbb{F})$, $L^{\epsilon}(\vec{A}) = \mathrm{Sym}^*(\vec{A})$.

The key lemma

Let $\vec{A} \in S_n^{\epsilon}(\mathbb{F})^m$, $D \in \mathrm{Sym}^*(\vec{A}) \subseteq \mathrm{Adj}(\vec{A})$, and $\dim(\mathrm{Adj}(\vec{A})) = \ell$.

Key idea

Consider D's action on $\mathrm{Adj}(\vec{A})$, e.g. D sends $E \in \mathrm{Adj}(\vec{A})$ to DE.

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- As a vector space, $\mathrm{Adj}(\vec{A}) \cong \mathbb{F}^\ell$, so $\tilde{D} \in \mathrm{M}_\ell(\mathbb{F})$.
- $\ker(\tilde{D}) = \mathrm{Ann}_r(D)$, the space of right annihilators of D.
- $\mathrm{im}(\tilde{D}) = D\mathrm{Adj}(\vec{A})$, the right ideal generated by D.
- D is full-rank if and only if $\tilde{D}$ is full-rank.

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Lemma (Key lemma)

If $\mathrm{Adj}(\vec{A})$ is semisimple, then for any non-full-rank $D \in \mathrm{Sym}^*(\vec{A})$, there exists $C \in \mathrm{Sym}^*(\mathcal{A})$ s.t. $C(\mathrm{Ann}_r(D)) \not\subseteq D\mathrm{Adj}(\vec{A})$.

In other words, $\tilde{C}(\ker(\tilde{D})) \not\subseteq \mathrm{im}(\tilde{D})$. (Simple rank increasing!)
And any linear basis of $\mathrm{Sym}^*(\vec{A})$ contains (at least) one such C.

The algorithm: without a mask

Suppose $\vec{A} \in S_n^{\epsilon}(\mathbb{F})^m$. Let $C_1, \dots, C_k$ be a basis of $\mathrm{Sym}^*(\vec{A})$. Let $F = \{\lambda_1, \dots, \lambda_{\ell+1}\} \subseteq \mathbb{F}$, where $\ell = \dim(\mathrm{Adj}(\vec{A}))$.

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If $\mathrm{Adj}(\vec{A})$ is semisimple, for a non-full-rank $D \in \mathrm{Sym}^*(\vec{A})$, we can choose $D' = C_i + \lambda_j D$ s.t. $\dim((C_i + \lambda_j D)\mathrm{Adj}(\vec{A}))$ is larger than $\dim(D\mathrm{Adj}(\vec{A}))$.

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When $\mathrm{Adj}(\vec{A})$ is not semisimple, but $\mathrm{Rad}(\mathrm{Adj}(\vec{A}))$ is efficiently computable, the same strategy works after modulo the radical.

- This new assumption holds for fields of characteristic 0 [Dickson] and finite fields [Rónyai].

The algorithm: with a mask

- Given $\vec{A} \in \mathrm{M}_n(\mathbb{F})^m$, $\vec{A} = E\vec{B}$ for some $\vec{B} \in S_n^\epsilon(\mathbb{F})^m$ and $E \in \mathrm{GL}_n(\mathbb{F})$.
- Let $D \in L^\epsilon(\vec{A})$. $D = D' E^{-1}$ for some $D' \in L^\epsilon(\vec{B}) = \mathrm{Sym}^*(\vec{B})$.
- Goal: compute $D' \mathrm{Adj}(\vec{B})$.

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- Goal: compute $D'\mathrm{Adj}(\vec{B})$.

- $\mathrm{Adj}(\vec{A}) = \mathrm{Adj}(\vec{B})$ because of the non-degeneracy condition and the projection to the second component.
  - $C^t(EA_i) = (EA_i)D$ if and only if $(E^tCE^{-t})A_i = A_iD$.
- $DL^{\epsilon}(\epsilon\vec{A}^t) = D'L^{\epsilon}(\vec{B})$.
  - $L^{\epsilon}(\epsilon\vec{A}^t) = L^{\epsilon}(\epsilon(E\vec{B})^t) = L^{\epsilon}(\epsilon\vec{B}^tE^t) = L^{\epsilon}(\vec{B}E^t) = EL^{\epsilon}(\vec{B})$.
- $DL^{\epsilon}(\epsilon\vec{A}^t)\mathrm{Adj}(\vec{A}) = D'L^{\epsilon}(\vec{B})\mathrm{Adj}(\vec{B}) = D'\mathrm{Adj}(\vec{B})$.

This means that we can work with $D'\mathrm{Adj}(\vec{B})$ without knowing the mask E!

Concluding remarks

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We also have algorithms when

- $\mathbb{F}$ is large enough without computing the radical;
- $\mathbb{F}$ is small.

Open questions:

- characteristic 2 fields?
- More examples of singular matrix spaces with no shrunk subspaces?

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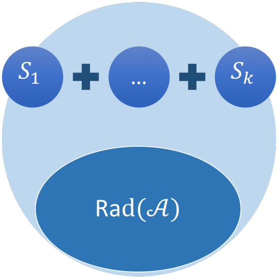
Structure of algebras

Let $\mathcal{A}$ be a finite dimensional associative algebra over $\mathbb{F}$. By Wedderburn et al., we have:

- $\mathrm{Rad}(\mathcal{A})$: the radical, e.g. the largest nilpotent ideal.
- $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$: semisimple, that is, isomorphic to a direct sum of simple algebras.
- $S_i \cong M(n_i, D_i)$: a full matrix algebra over D_i, a division algebra over $\mathbb{F}$.



Structure of $*$-algebras

Let $* : \mathcal{A} \to \mathcal{A}$ be an involution, e.g. an anti-automorphism such that $\forall a \in \mathcal{A}$, $(a^*)^* = a$. By Albert et al., we have:

- $\mathrm{Rad}(\mathcal{A})$ is invariant under $*$: $*$ induces an involution on $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$.
- Recall that $S_i \cong M(n_i, D_i)$.
    1. (Exchange type) $S_i^* = S_j$, $i \neq j$. Then $S_i \cong S_j$, and $(a, b)^* = (b, a)$, $(a, b) \in S_i \oplus S_j$.
    2. (Classical type) $S_i^* = S_i$. There is a classical form $F \in M(n_i, D_i)$, such that $A^* = F^{-1} A^t F$.