Optimal Lower Bounds for Distributed and Streaming Spanning Forest Computation

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Joint work with Jelani Nelson

Warm-up

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what if we allow edge **deletions**?

Fully dynamic spanning forest

Maintain a dynamic graph on n vertices, supporting

- edge insertions,
- edge deletions, and
- spanning forest queries

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Theorem (Ahn, Guha, McGregor'12)

... solvable using $O(n \log^3 n)$ bits of space with error probability $1/\mathrm{poly}(n)$.

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why two more?

Main result I

Theorem (This paper)

Any data structure for fully dynamic spanning forest with error probability δ must use $\Omega(n \log(n/\delta) \log^2 n)$ bits of memory, for any $2^{-n^{0.99}} < \delta < 0.99$.

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δ is a constant $\implies \Omega(n \log^3 n)$ bits of space:

need *exactly two more* log factors!

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(compute a global function given small "sketches" of "local information")

AGM sketch for simultaneous communication

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Trivial: $\Omega(\log n)$ since the referee has to learn $\Omega(n \log n)$ bits

Main result II

Theorem (This paper)

Any simultaneous communication protocol for spanning forest with error probability 0.99 must use $\Omega(\log^3 n)$ bits of communication on average.

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Open: higher lower bounds when error probability δ is lower?

Graph sketching for spanning forest

[AGM'12] designed a (randomized) linear sketch:

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  each $S_i(G)$ has $O(\log^2 n)$ dimensions,
  and it is computed from the neighborhood of vertex i
- $S(G)$ determines a spanning forest with probability $1 - 1/n^c$

Streaming algorithm

Store $S(G)$ in memory:

- update: $S(G \pm (u, v)) = S(G) \pm S((u, v))$
- at end of stream: $S(G)$ determines a spanning forest w.h.p.

Use $O(n \log^3 n)$ bits of space

Communication protocol

Given graph G:

- Player i computes $S_i(G)$, and sends it to referee
- referee concatenates all $S_i(G)$, obtains $S(G)$
- referee outputs a spanning forest w.h.p.

Use $O(\log^3 n)$ bits of communication per player

Simultaneous communication complexity of spanning forest

Recall...

An *n*-vertex graph is given to *n* players with shared randomness:

- each player only sees one vertex and its neighborhood
- each player sends a message to a referee
- referee outputs a spanning forest w.p. $1 - \delta$

Goal: prove an average player must send $\Omega(\log^3 n)$ bits for constant δ

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An n-vertex graph is given to n players with shared randomness:

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Goal: prove some player must send $\Omega(\log^3 n)$ bits for $\delta = 1/n^c$

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Starting point: Universal Relation $\mathrm{UR}^{\supset}$...

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shared random bits...

Alice: $S \subseteq [n]$ $\xrightarrow{\mathbf{M}}$ Bob: $T \subset S$

output any $x \in S \setminus T$

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Theorem (KNPWWY'17)

*For failure probability $\delta > 2^{-n^{0.99}}$, the optimal length of **M** is $\Theta(\log(1/\delta)\log^2 n)$.*

Universal Relation $\mathrm{UR}^{\supset}$

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$\text{Alice: } S \subseteq [n] \xrightarrow{\mathbf{M}} \text{Bob: } T \subset S$

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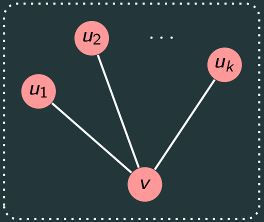
Theorem (KNPWWY'17)

For failure probability $\delta > 2^{-n^{0.99}}$, the optimal length of $\mathbf{M}$ is $\Theta(\log(1/\delta)\log^2 n)$.

In particular, $1/n^c$ failure probability, optimal length is $\Theta(\log^3 n)$.

Connection to $\mathsf{UR}^{\supset}$



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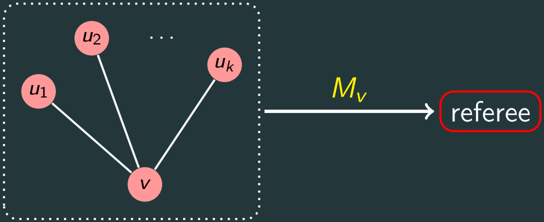

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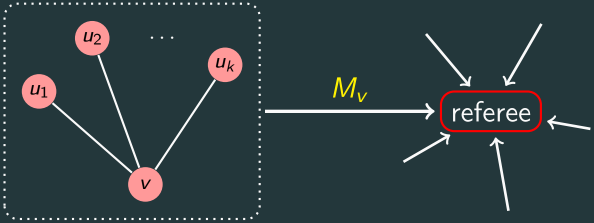

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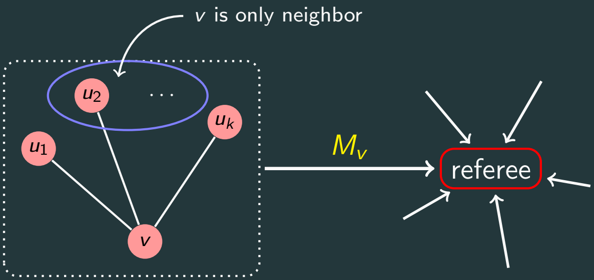

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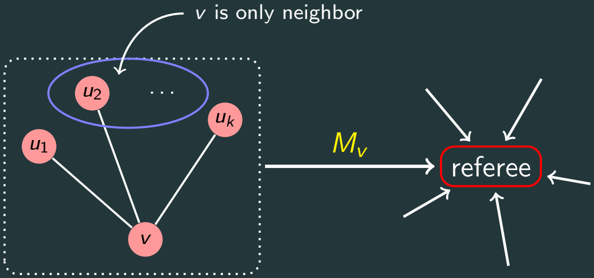

S: neighbors of v T: vertices that v is only neighbor

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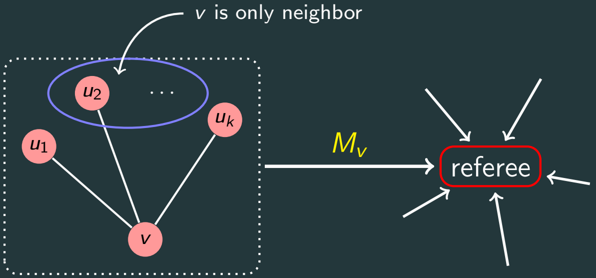

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Referee has to find some element in $S \setminus T$.

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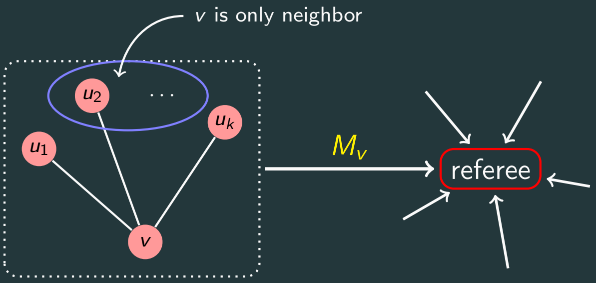

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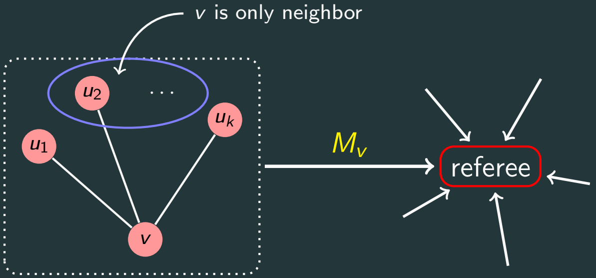

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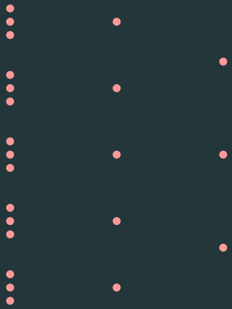
Why not already an $\Omega(\log^3 n)$ LB? M_{u_1} may also reveal (v, u_1)...

Hard instances



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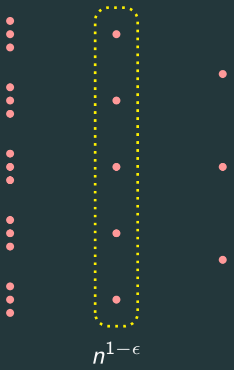

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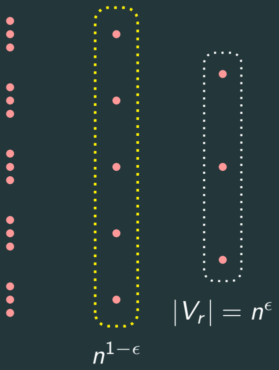

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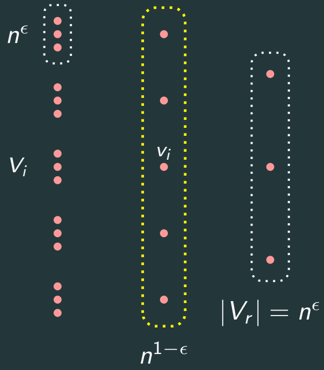

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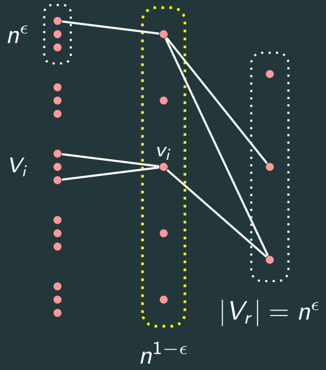

vertices randomly permuted

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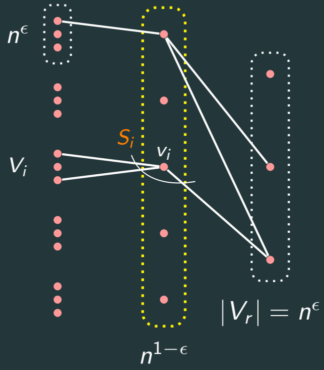

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For vertex v_i, its neighbors encode set S_i

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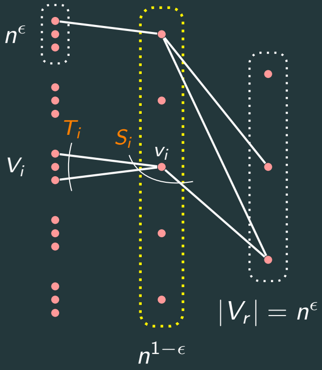

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For vertex v_i, its neighbors encode set S_i, its neighbors on the left encode set T_i.

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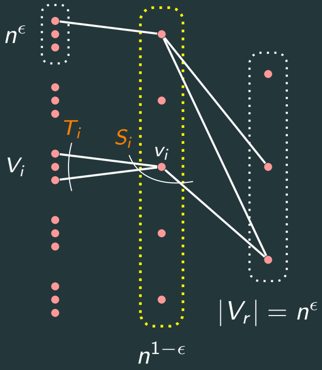

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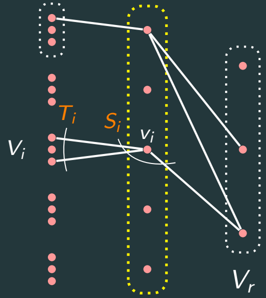
Spanning forest contains an edge between v_i and V_r.

Hard distribution

Generating hard instances:

1. Fix $\{v_i\}$ arbitrarily, randomly partition the rest into $\{V_i\}$, V_r;



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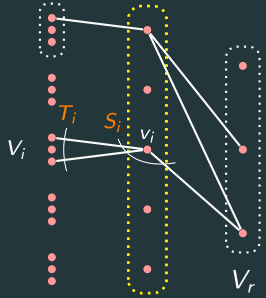
1. Fix $\{v_i\}$ arbitrarily, randomly partition the rest into $\{V_i\}$, V_r;
2. For each v_i, generate S_i, T_i from hard distribution for $\mathrm{UR}^{\supset}$;



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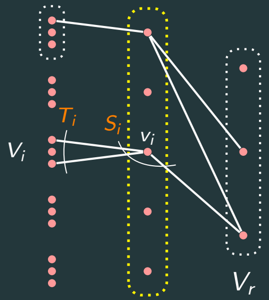
3. Connect each v_i to $|T_i|$ random vertices in V_i;
4. Connect each v_i to $|S_i \setminus T_i|$ random vertices in V_r.



Reduction from $\mathrm{UR}^{\supset}$

Make a reduction from $\mathrm{UR}^{\supset}$, main idea to solve $\mathrm{UR}^{\supset}$:

embed input (S, T) into one of (S_i, T_i),

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3. Low communication cost and preserve success probability

Solving $\mathrm{UR}^{\supset}$

Given (S, T) over universe $[n^{\epsilon}]$, generate a random graph G:

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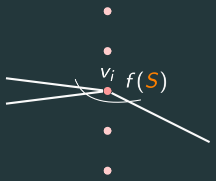

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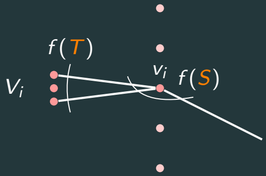

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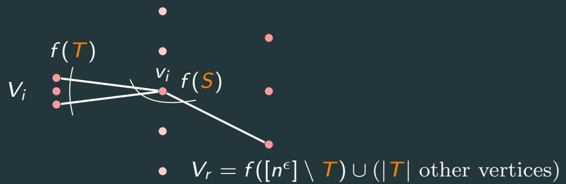

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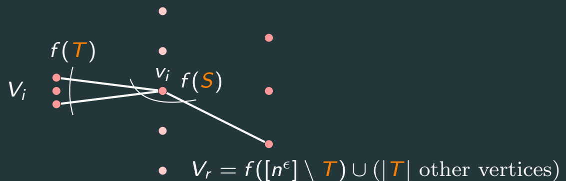
5. Randomly partition other vertices into $V_1, \ldots, V_{i-1}, V_{i+1}, \ldots$, sample the neighborhoods of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots$



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Distribution of G is the hard distribution.

Let u be one v_i's neighbor in V_r, then $f^{-1}(u) \in S \setminus T$.

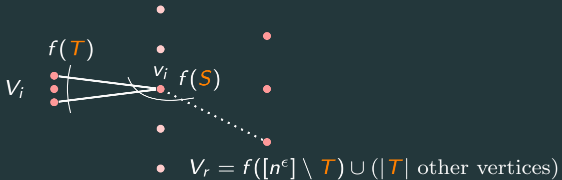
$\mathrm{UR}^{\supset}$ **protocol**

Given (S, T) over universe $[n^{\epsilon}]$

A: send M_{v_i} based on $f(S)$

B: analyze the distribution of G *conditioned on* f, T, M_{v_i}

B: find $u \in V_r$ that is a neighbor of v_i with the highest prob., output $f^{-1}(u)$



Analyzing the protocol

The protocol for $\mathrm{UR}^{\supset}$ has

- communication cost $|M_{v_i}|$, and
- failure probability $\leq \delta + 1/n^{0.1}$.

By [KNPWWY'17], $|M_{v_i}| \geq \Omega(\log(1/\delta)\log^2 n)$

($\Omega(\log^3 n)$ lower bound when $\delta = 1/n^c$)

Open question

Lower bounds for simultaneous communication when error probability is small? $\Omega(\log(n/\delta)\log^2 n)$?

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Proving the same lower bounds for maintaining connected components? and for connectivity: "if the whole graph is connected"?

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Thank you!