NETWORK CODING IN UNDIRECTED GRAPHS IS EITHER VERY HELPFUL OR NOT HELPFUL AT ALL

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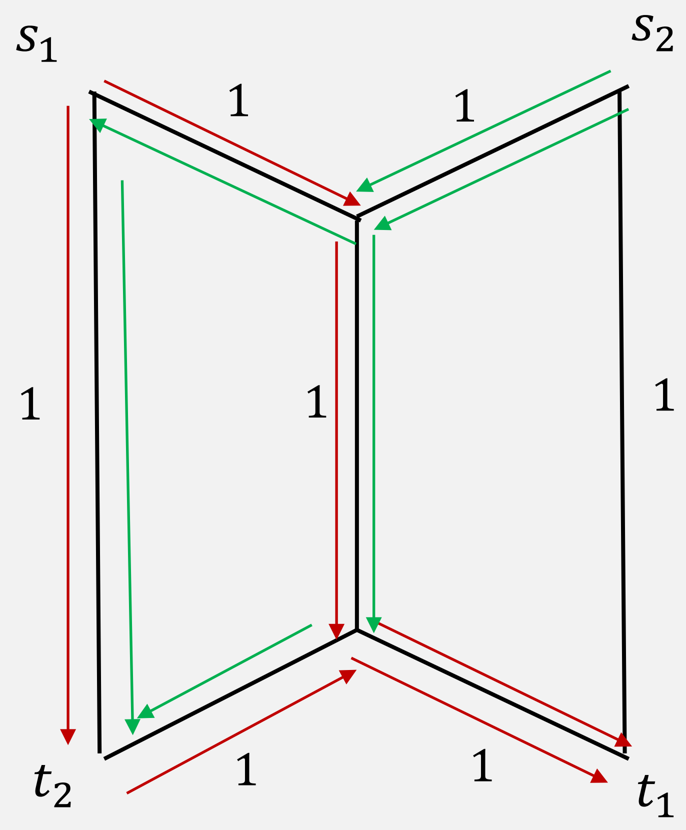
MULTICOMMODITY FLOW

- Graph $G = (V, E)$
- Capacity function $c: E \rightarrow R^+$
- Set I of k commodities: $\{(s_i, t_i), i \in [k]\}$
- Rate is r:
  - Flow between source-sink pair is atleast r
  - Total flow through an edge is upper bounded by its capacity
- $MCF(G)$ denotes the multicommodity flow rate

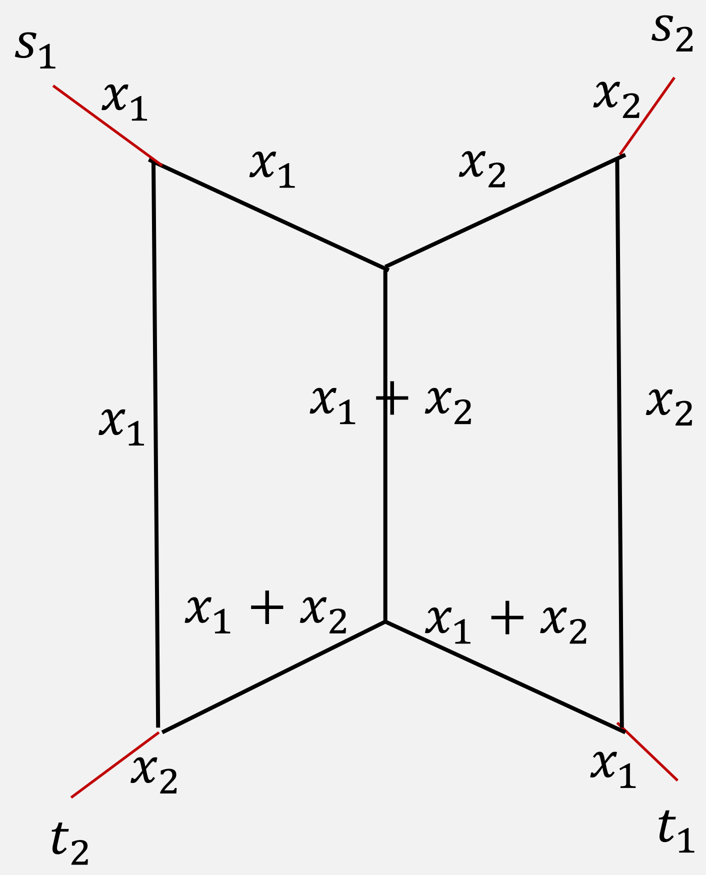


FLOWS OF PACKETS

- No re-encodings at intermediate nodes
- The Question: Can we get better information rate if the commodities are bits and we use bit tricks on them?

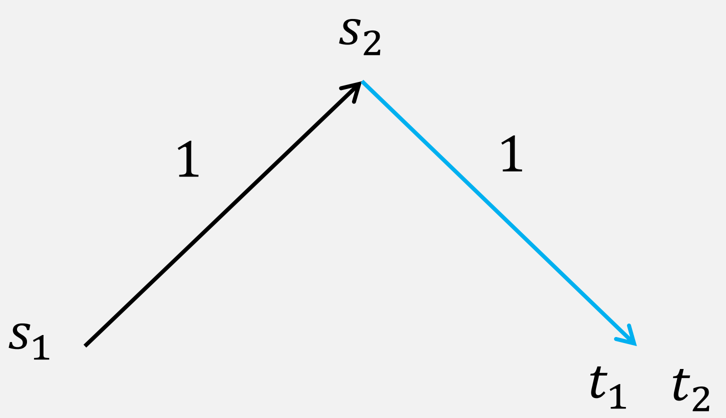
NETWORK CODING

- First introduced in [ACLY '00]
- $M = \{M_i\}_{i\in[k]}$ is the messages by sources
- Each edge has $f: M \to \Delta(e)$ (alphabet for e)
  - Function of alphabets on incoming edges
  - Entropy(e) is upper bounded by capacity
- Each sink edge t_i carries M_i



NETWORK CODING RATE

- $NC(G)$ denotes the network coding rate
  - $r = \lim_{b \to \infty} (\frac{\max r_b}{b})$ (All capacities are multiplied by b)



$r_1 = 0$

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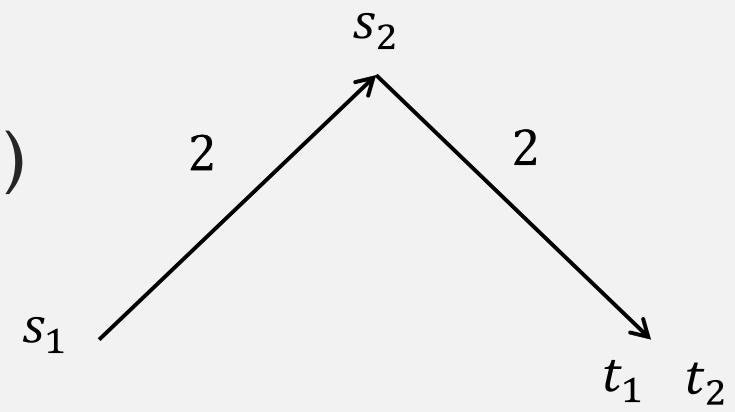
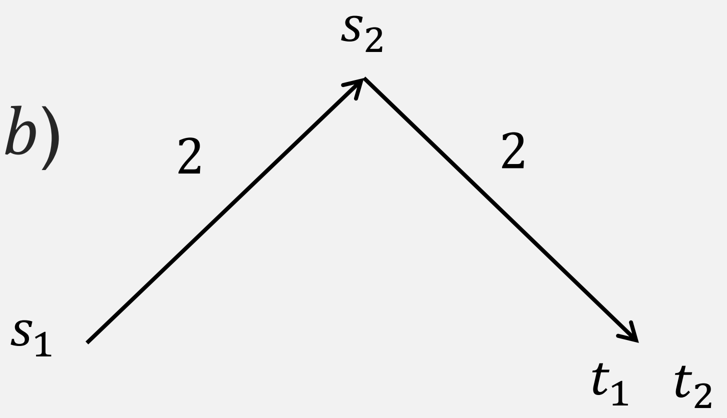


$r_2 = 1, NC = \frac{1}{2}$

NETWORK CODING RATE

- $NC(G)$ denotes the network coding rate
  - $r = \lim_{b \to \infty} (\frac{\max r_b}{b})$ (All capacities are multiplied by b)
- Decidable?
- $Gap(G) = NC(G)/MCF(G)$
- Is there a G with $Gap(G) > 1$?



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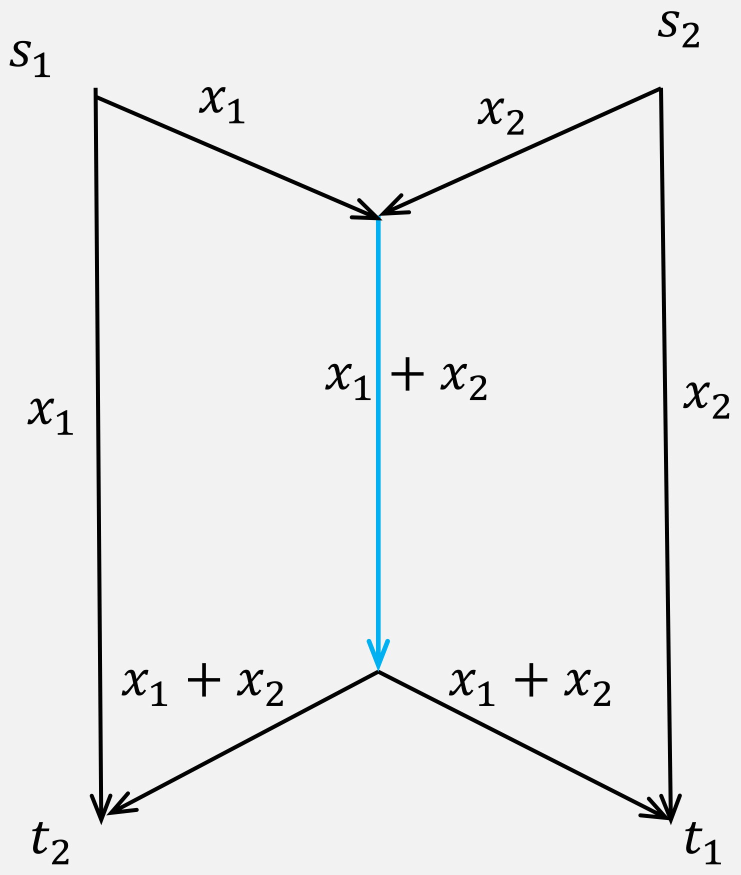
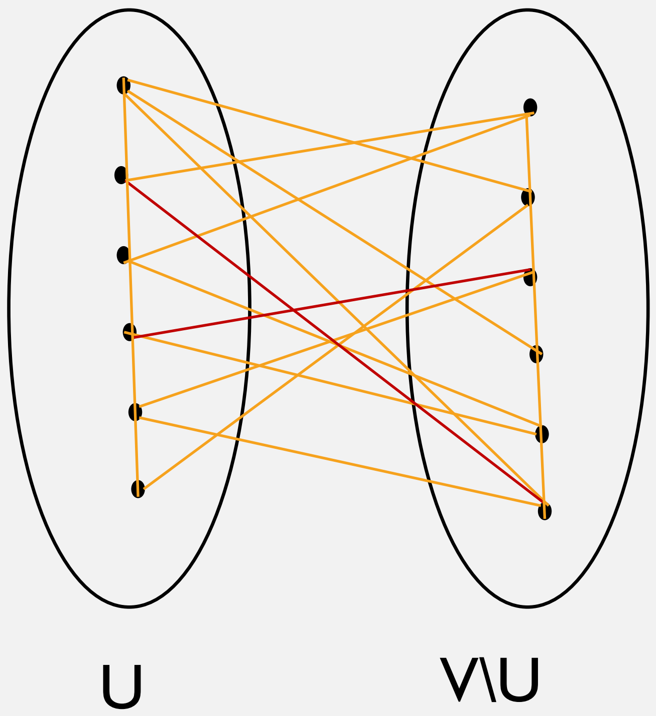
DIRECTED GRAPHS

- Yes! [HKL '04] [LL '04]
- NC/MCF gap can be as large as $O(|G|)$



SPARSITY BOUND FOR UNDIRECTED

- $U \subseteq V, Sparsity(U, V \setminus U) = \frac{Capacity(U, V \setminus U)}{Demand(U, V \setminus U)}$



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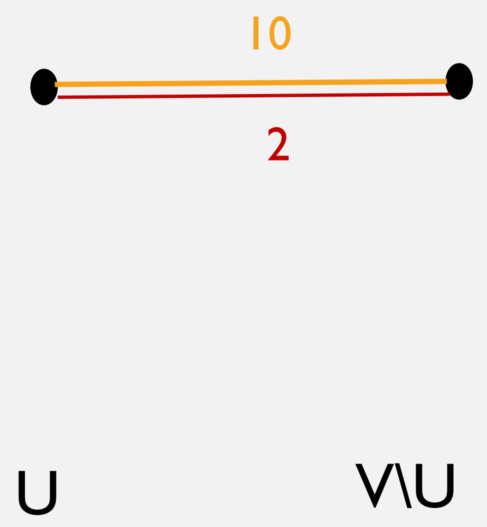

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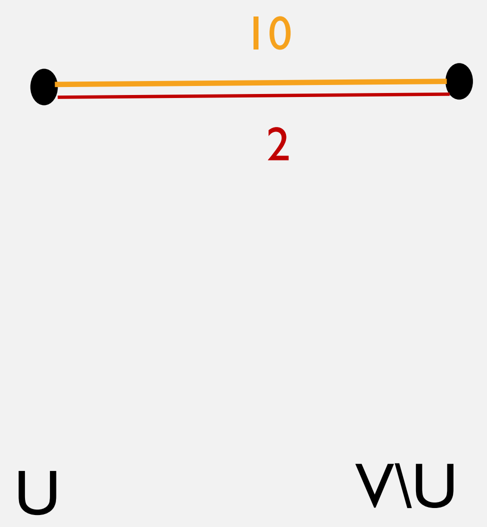
- $Sparsity(G) = \min_{U \subseteq V} Sparsity(U, V \backslash \mathrm{U})$



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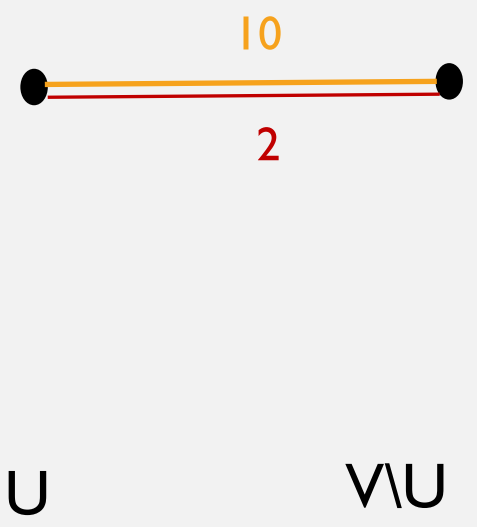
- $Sparsity(G) = \min_{U \subseteq V} Sparsity(U, V \backslash U)$
- $MCF(G) \leq NC(G) \leq Sparsity(G)$
- $\frac{Sparsity(G)}{O(\log |G|)} \leq MCF(G) \leq Sparsity(G)$ [LR '99]

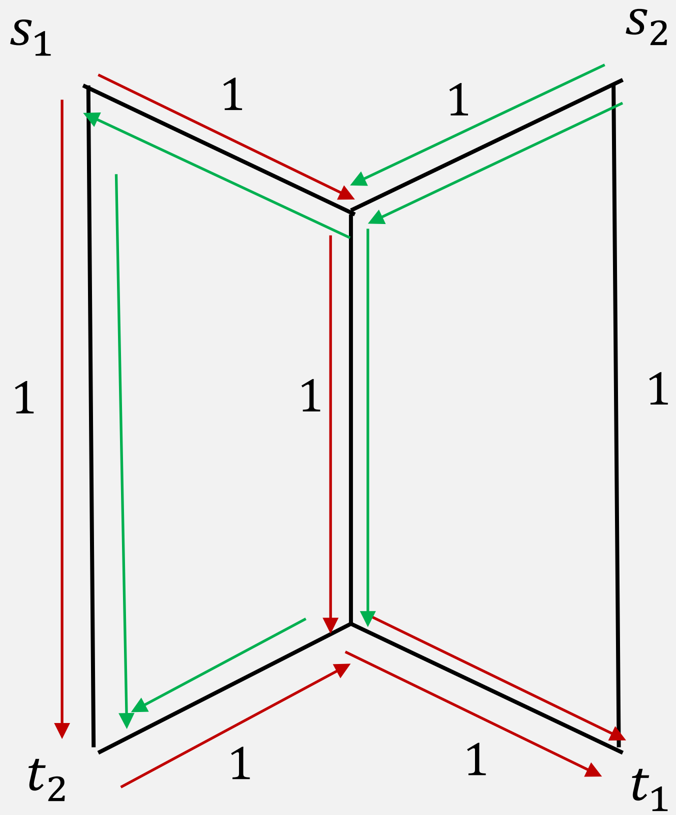
Expanders

Information Upper Bound



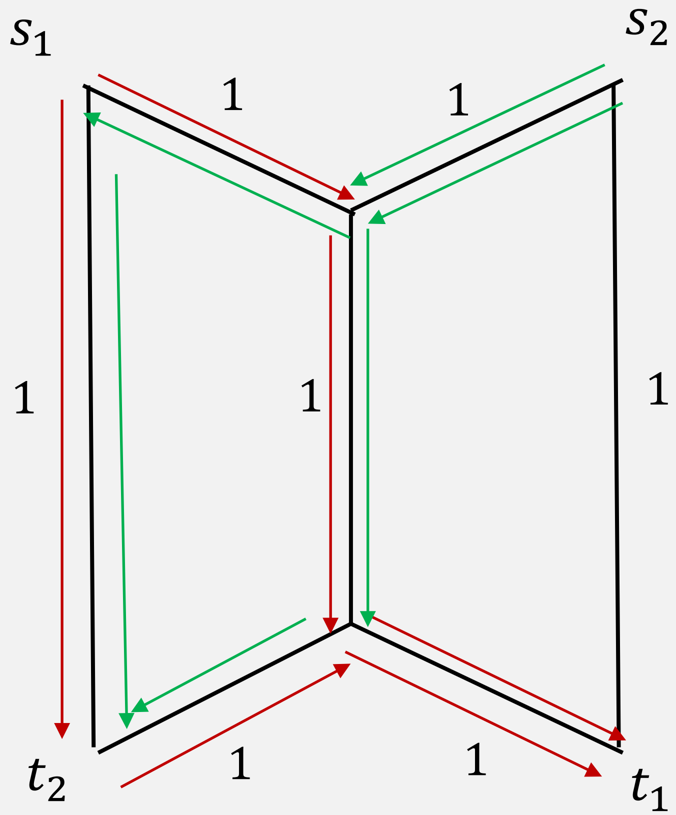
MAXIMUM GAP IN UNDIRECTED GRAPHS

- $\frac{NC(G)}{O(\log|G|)} \leq \frac{Sparsity(G)}{O(\log|G|)} \leq MCF(G) \leq NC(G)$
- Maximum gap can be $O(\log|G|)$



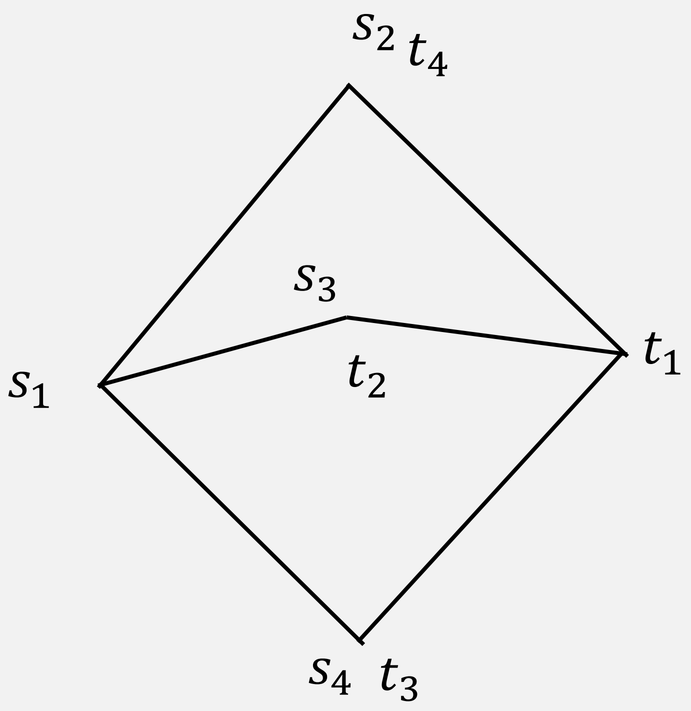
MAXIMUM GAP IN UNDIRECTED GRAPHS

- $\frac{NC(G)}{O(\log|G|)} \leq \frac{Sparsity(G)}{O(\log|G|)} \leq MCF(G) \leq NC(G)$
- Maximum gap can be $O(\log|G|)$
- Li and Li **conjectured** that $\boldsymbol{MCF(G) = NC(G)\ \forall G}$ [LL '04]
  - Coding gives no advantage



RELATED WORKS

- [SYC '03] [KS '03] [K '03] [JVY '05] [KS '06] [HKL '06] produced techniques for lower and upper bounding NC



OUR RESULT

- Either the conjecture is **true** or it must be nearly **'completely false'**

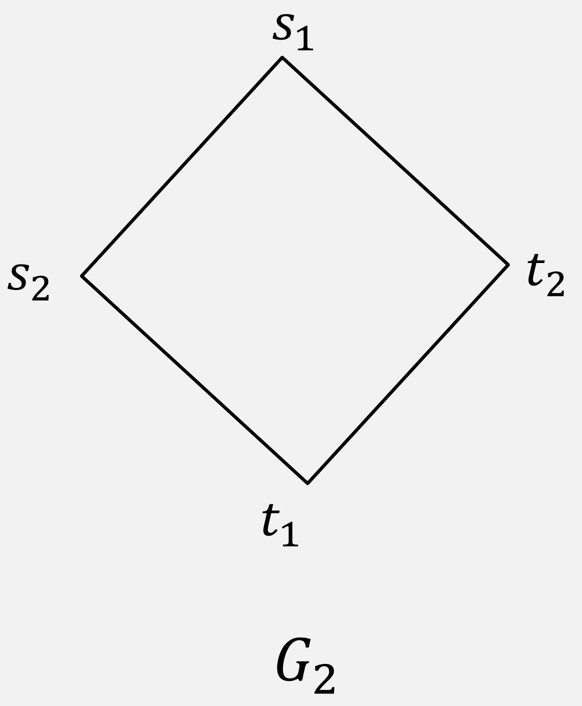
▶ **Theorem 1.** *Given a graph G that achieves a gap of* $1+\epsilon$ *between the multicommodity flow rate and the network coding rate, we can construct an infinite family of graphs* $\widetilde{G}$ *that achieve a gap of* $O\left(\log|\widetilde{G}|\right)^c$ *for some constant* $c < 1$ *that depends on the original graph* G.

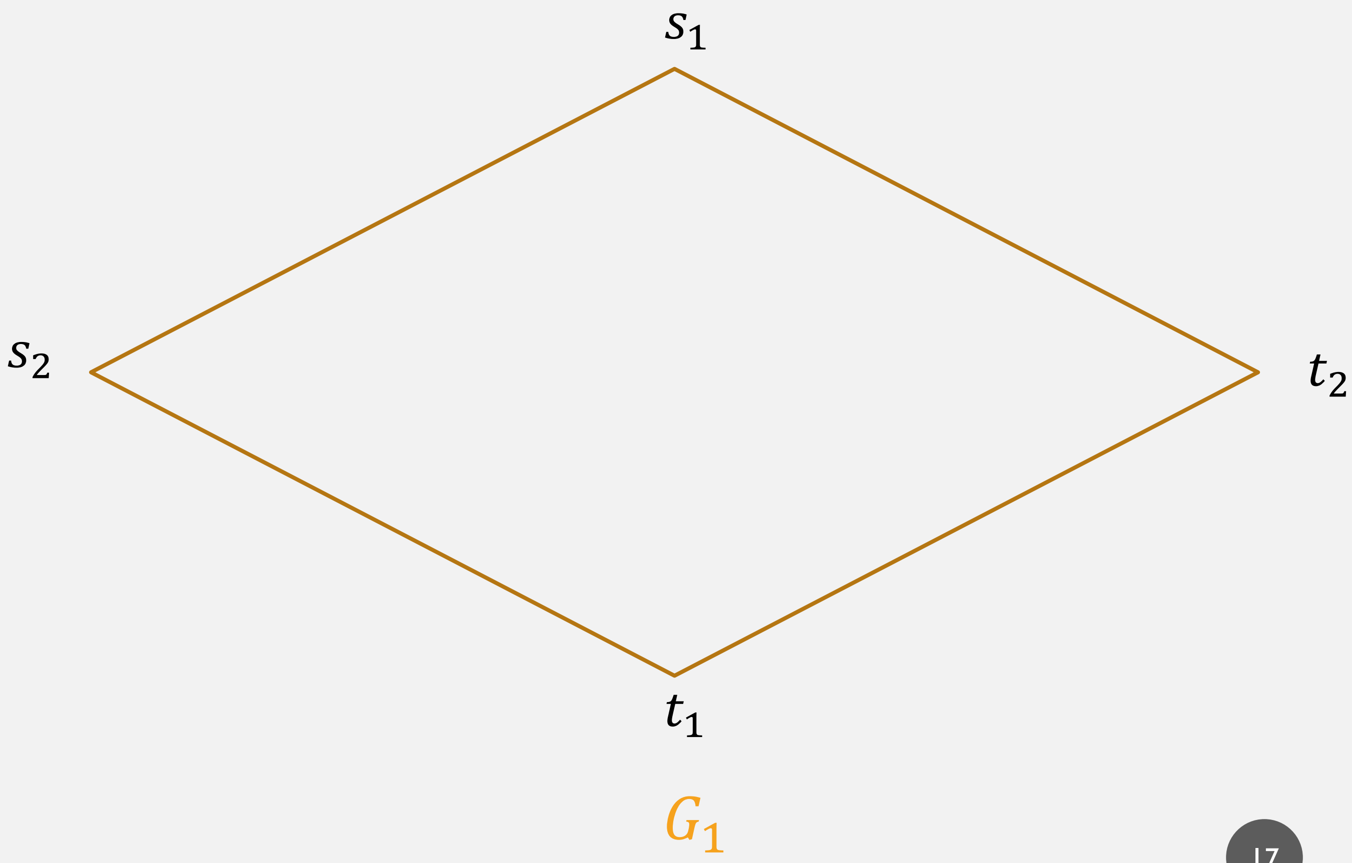
BUILDING BLOCK

- Given two graphs G_1 and G_2 with gaps $(1+\epsilon_1)$ and $(1+\epsilon_2)$ respectively, we construct a new graph G with gap $(1+\epsilon_1)(1+\epsilon_2)$ while keeping a check on size of G
- Apply this construction repeatedly on the starting graph with the gap

GRAPH TENSOR

- Replace each edge of G_1 by a source-sink pair of G_2





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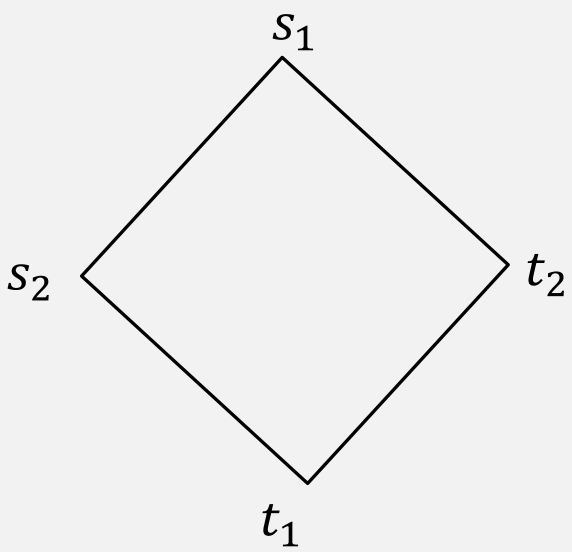
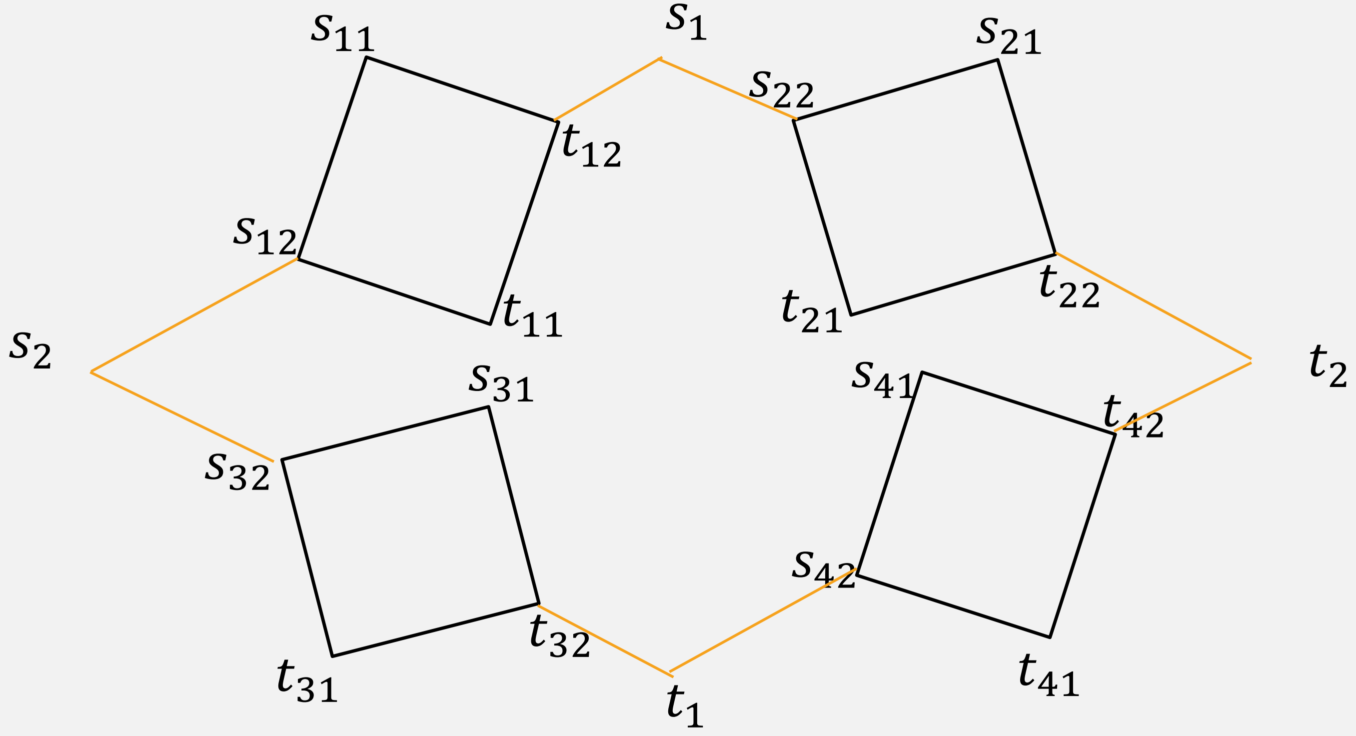


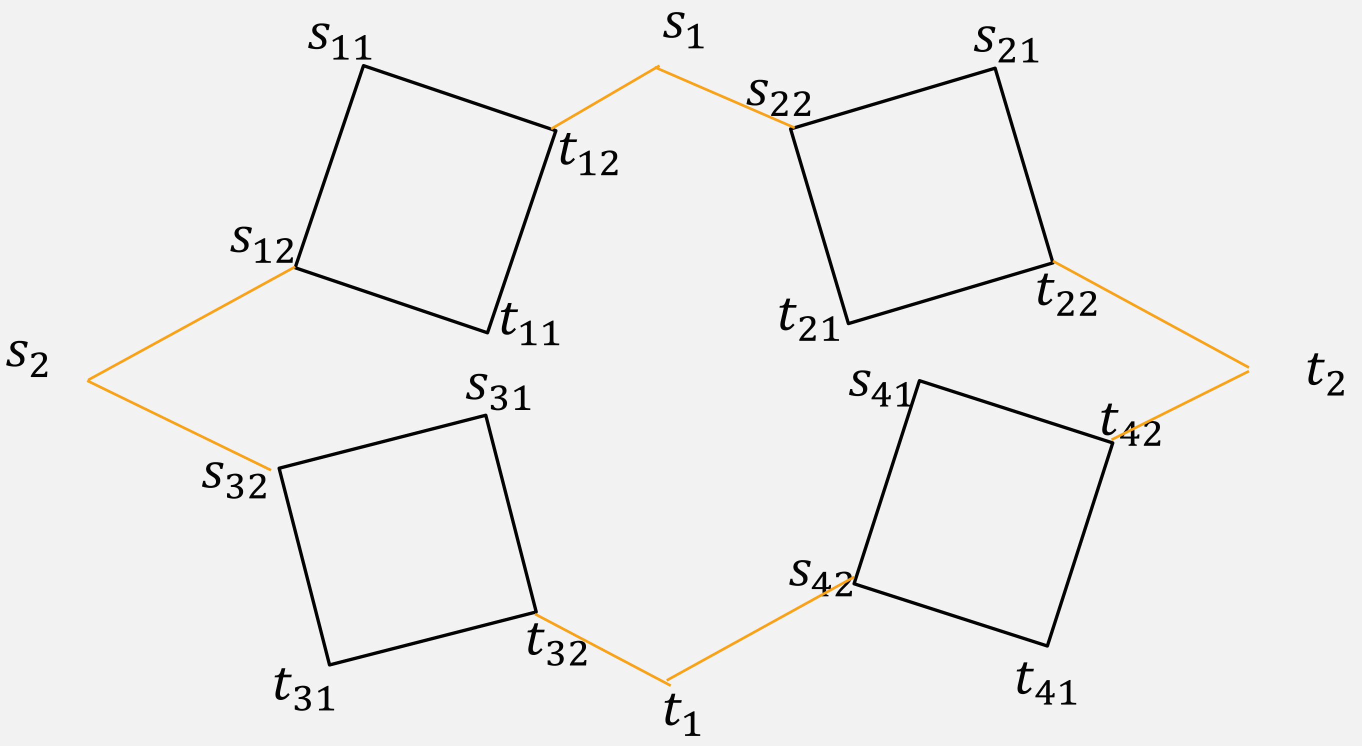
G_2



$G_1 \otimes G_2$ Gadget

GRAPH TENSOR

- Keep the source/sink pairs of G_1 and edges of G_2



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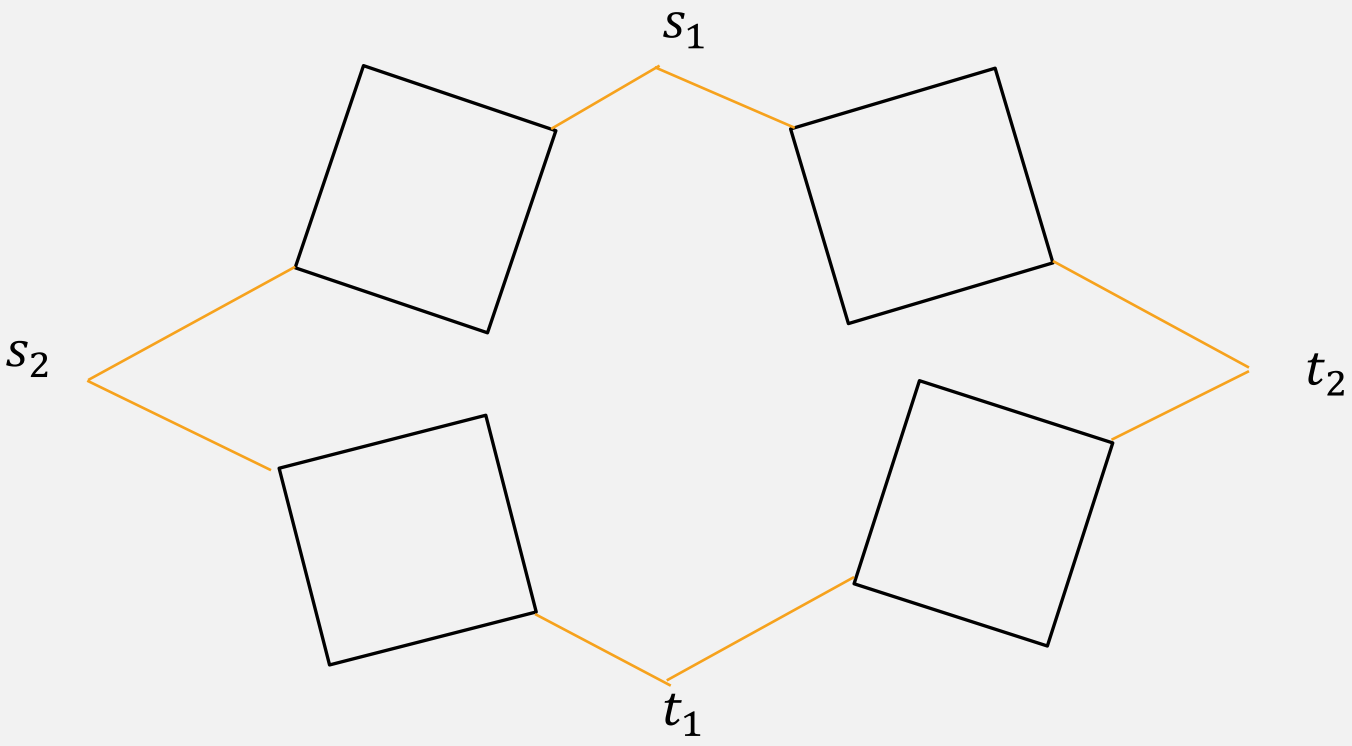


$G_1 \otimes G_2$ Gadget

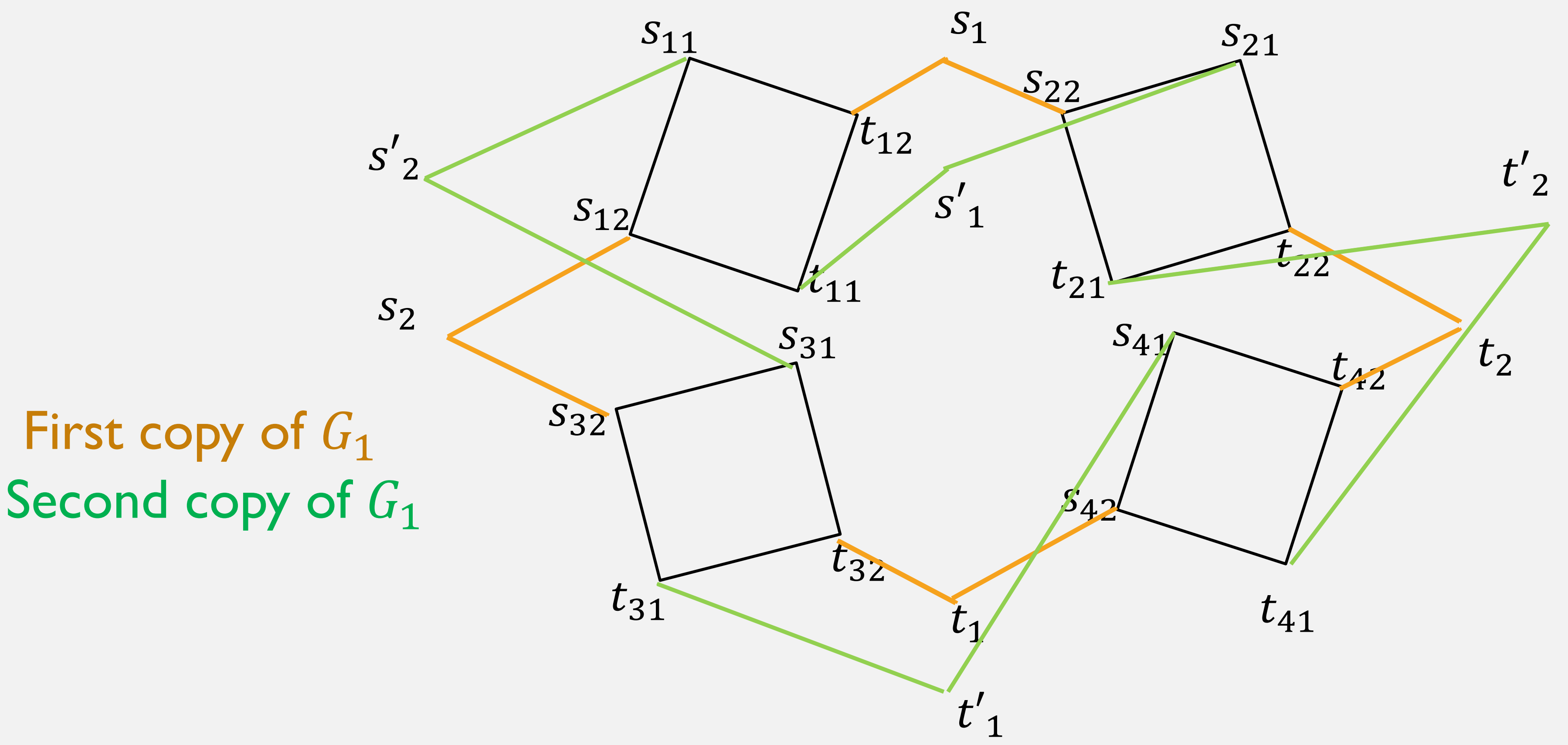
IDEA

- Idea: Effective capacity seen by G_1 under network coding is greater than that seen under flows
- Information transferred grows linearly with capacity
  - Gaps should multiply

NOT THERE YET

- For G_2, there is a gap only when all source-sink pairs send flows simultaneously
- We have multiple copies of G_1 and each source-sink pair in a copy of G_2 replaces an edge in a different copy of G_1

UPDATED GRAPH TENSOR



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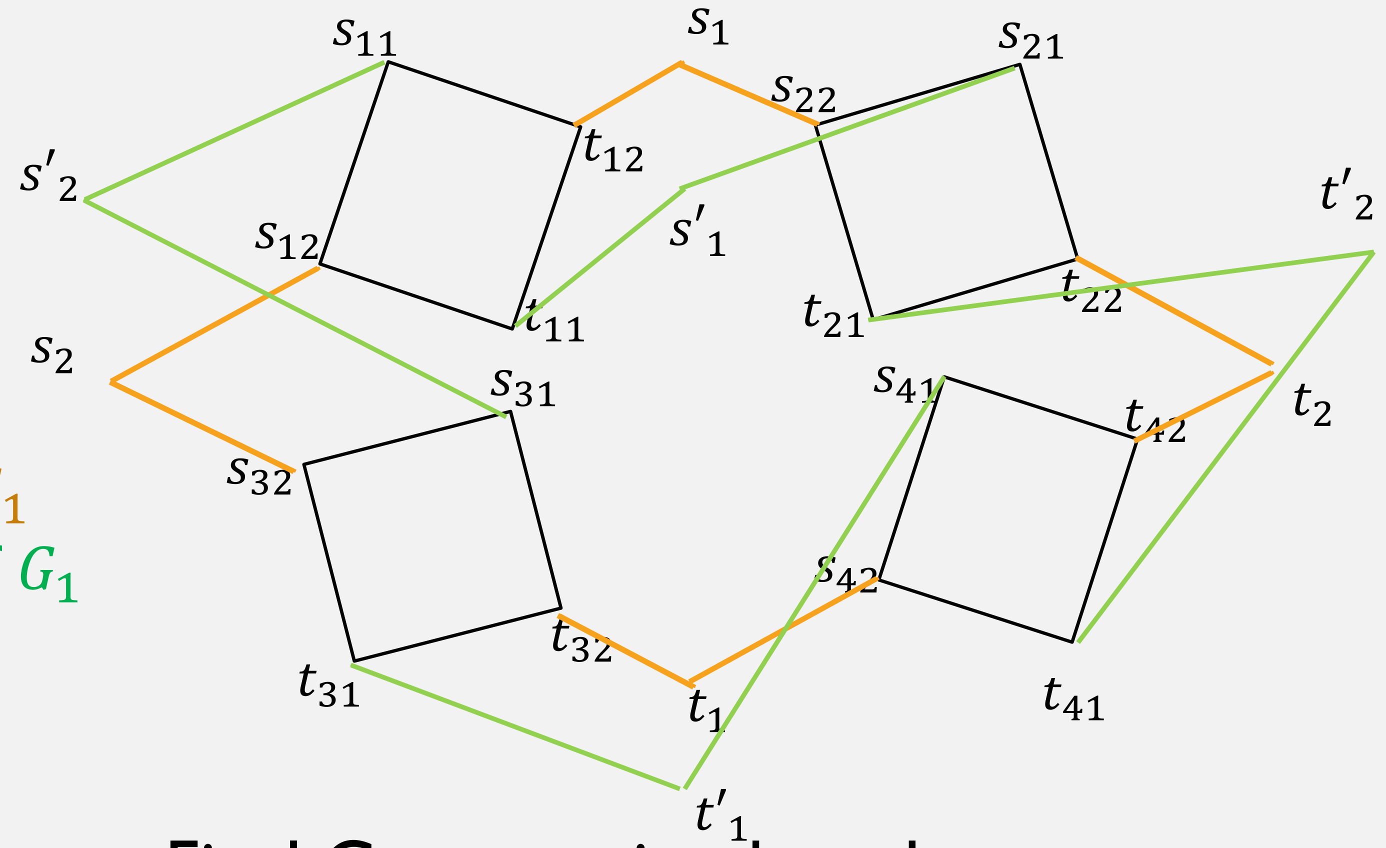

First copy of G_1

Second copy of G_1

Not the final tensor: Final Construction based on high girth bipartite graphs

MAIN THEOREM

- Start with a graph $G_0 = G$ with gap $(1 + \epsilon)$

▶ **Theorem 2.** *Given a graph G of size n with a gap of $1 + \epsilon$ between the multicommodity flow rate and the network coding rate, we can create another graph G' of size n^{c^2} and a gap of $(1 + \epsilon)^2$, where c depends on the diameter of the graph G.*

ITERATIVE TENSORING

- For iteration $j, G_j = G_{j-1} \otimes G_{j-1}$ (Applying theorem 2 to G_{j-1} to get G_j)
- Gap $= (1+\epsilon)^{2^j}$
- Size grows like $n^{c^{2^j}}$
- Gap grows as $O\left(\log|G_j|\right)^{c_1}$ where c_1 is a constant < 1!

OPEN PROBLEMS

- Proving/ Disproving the Li and Li conjecture
  - Even for linear codes?
- Computing optimal network coding for directed graphs

ENTROPIC REGION [CG '08]

- n Random variables $X_1, X_2, \ldots, X_n$ (joint distribution)
- Entropic vector is $2^n - 1$ dimensional vector with S^{th} coordinate holding $H(X_{i_1}, X_{i_2}, \ldots, X_{i_k})$ where $S = \{i_1, i_2, \ldots, i_k\} \subseteq [n]$
  - $n = 2,\ [H(X_1), H(X_2), H(X_1 X_2)]$

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- Entropic region is the set of all such vectors
  - $H(X_1) \leq H(X_1 X_2) \leq H(X_1) + H(X_2)$

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- Entropic region is the set of all such vectors
  - $H(X_1) \leq H(X_1X_2) \leq H(X_1) + H(X_2)$
- Given a $2^n - 1$ dimensional vector, is it ϵ-close to a vector in entropic region? (Even decidability)

THANKS