Recent Structure Lemmas for Depth-Two Threshold Circuits

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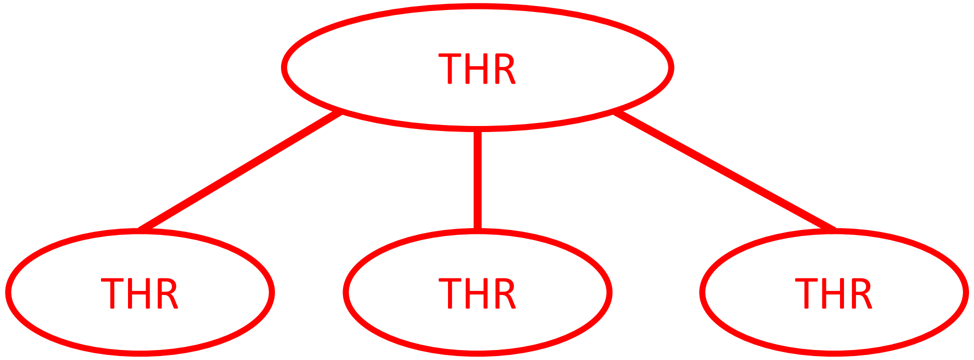
THR∘THR Circuits

THR gates : $f(x) = [w \cdot x \geq t]\ w \in Z^n, t \in Z.$

MAJ gates : when w_i's and t are bounded by poly(n).

We can also define
$THR \circ MAJ$
$MAJ \circ THR$
$MAJ \circ MAJ$

THR∘THR:



THR∘THR Circuits

Exponential Lower Bound are known for

$MAJ \circ MAJ$ [Hajnal-Maass-Pudlák-Szegedy-Turán'93]

$MAJ \circ THR$ [Nisan'94]

$THR \circ MAJ$ [Forster-Krause-Lokam-Mubarakzjanov-Schmitt-Simon'01]

NEXP:
Non-deterministic
Exponential Time.

Frontier Open Question: *Is* $NEXP \subseteq THR \circ THR$?

Potential Approaches in this talk.

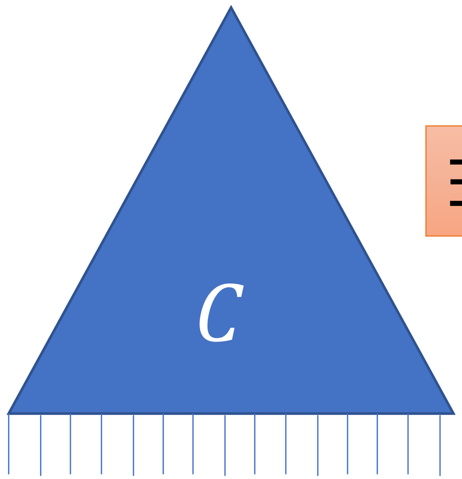
Motivation

R. Williams' algorithmic approach to lower bounds:
Lower Bounds for $\boldsymbol{C}$ from Non-trivial Algorithms for $\boldsymbol{C}$.
(some subtleties in depth increase of C, but turns out we can handle it!)

Natural Question:
How hard is **algorithmic analysis** of $THR \circ THR$ circuits?
How does it compare to $THR \circ MAJ$ or $MAJ \circ MAJ$?
*(Spoiler): They're equally hard/easy… **Counterintuitive!***

Algorithmic Questions

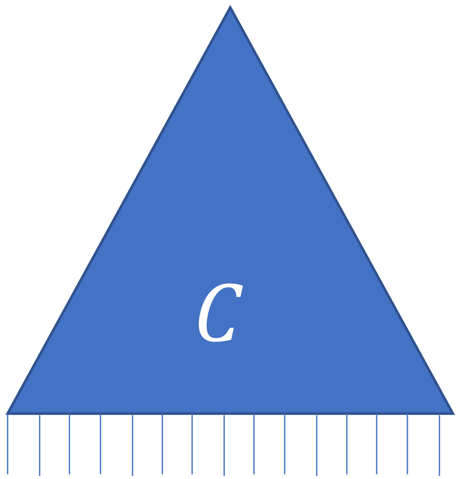
C-SAT



$\exists$ x s.t. $C(x) = 1$?

$\exists x$?

C-CAPP



Estimate quantity $\Pr_{x \sim U_n}[C(x) = 1]$, with additive error ε

$x \sim U_n$

Algorithmic Questions

C-SAT

C-CAPP

***Define** non-trivial algorithms:*

1. Non-trivial SAT: $2^n/n^{\omega(1)}$ time algorithm for SAT.
2. Non-trivial CAPP: $2^n/\mathrm{n}^{\omega(1)}$ time for CAPP with error $\varepsilon = 1/poly(n)$.

ntity

1],

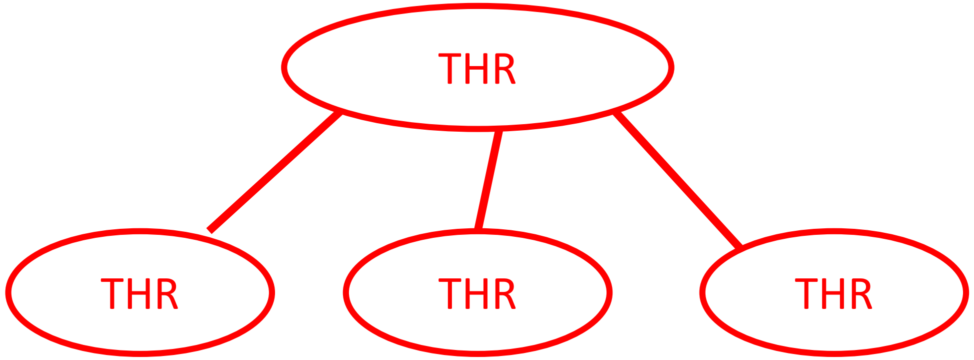
error ε

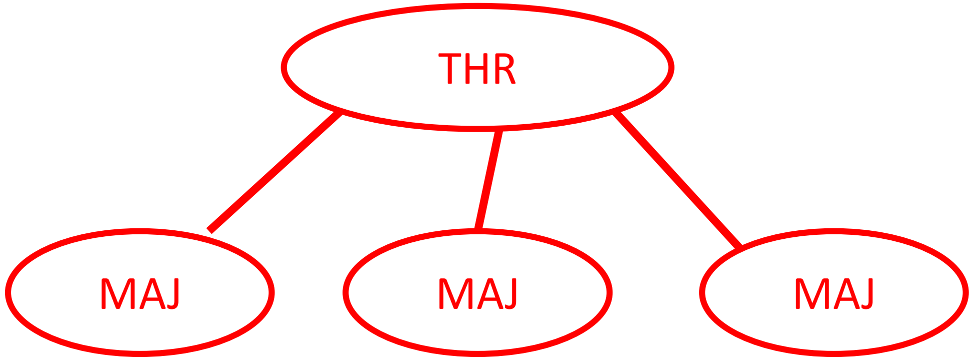
$\exists x$?

$x \sim U_n$

Algorithmic Equivalence I

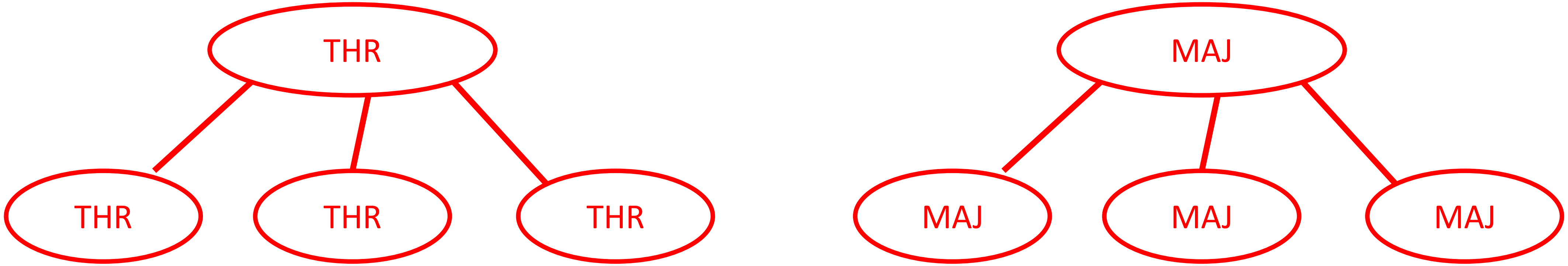
Poly-size $THR \circ THR$ and $THR \circ MAJ$ are **equivalent** for Non-Trivial SAT Algorithms!





Algorithmic Equivalence II

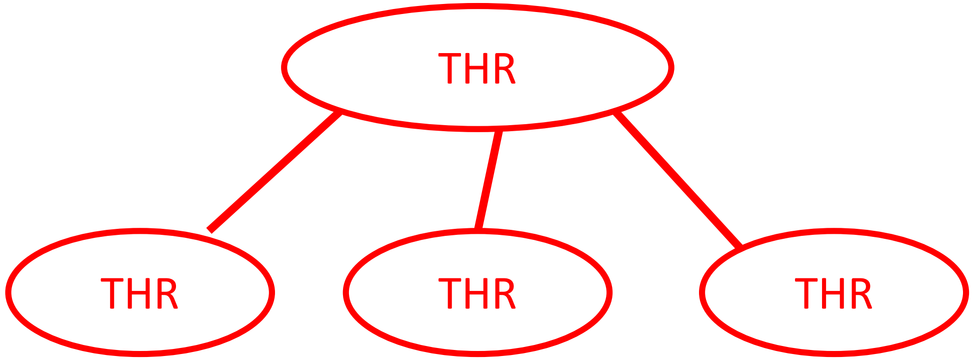
Poly-size $THR \circ THR$ and $MAJ \circ MAJ$ are **equivalent** for Non-Trivial CAPP Algorithms!

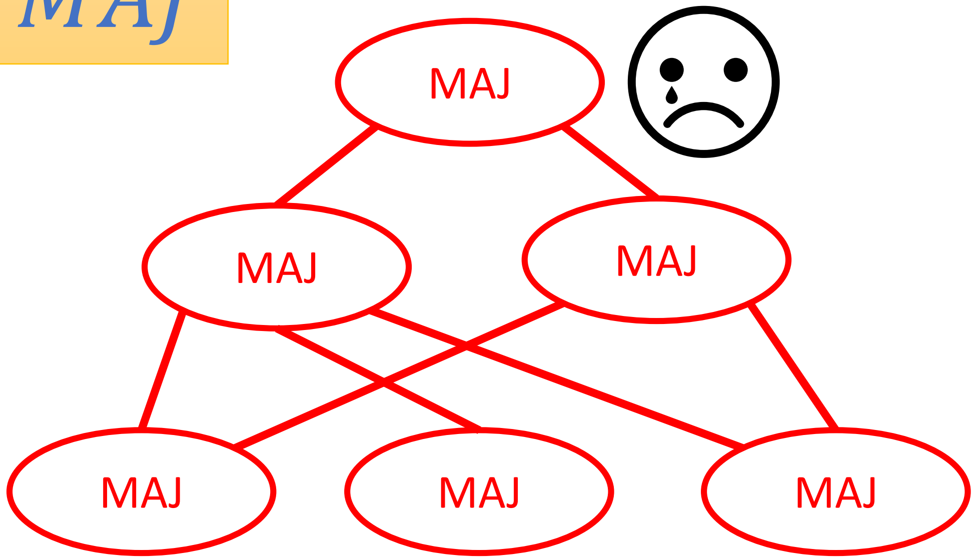


Motivation: New Structure Lemmas

[Goldmann-Hastad-Razborov'92]:

$THR \circ THR \subseteq MAJ \circ MAJ \circ MAJ$





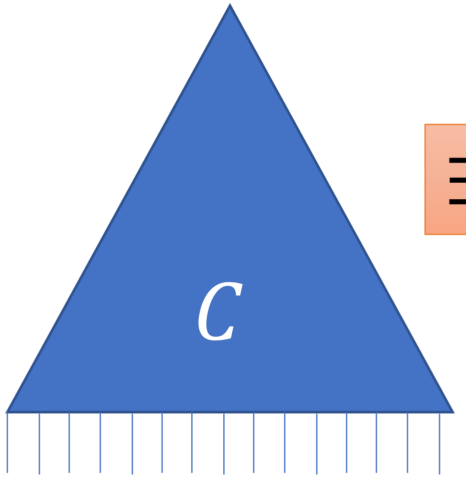
Stuck!?!

How do we use $(MAJ \circ MAJ)$ – SAT to solve SAT for $MAJ \circ MAJ \circ MAJ$?

Motivation: New Structure Lemmas

What if we had a top $\boldsymbol{OR}$ gate? e.g., $OR \circ MAJ \circ MAJ$

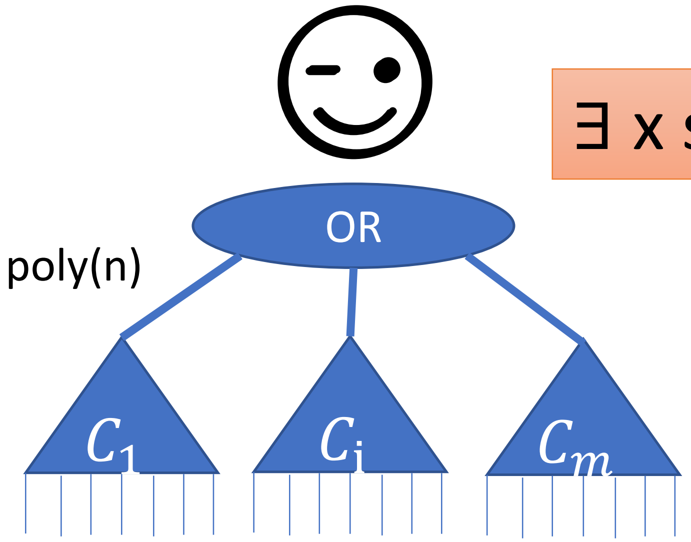
C-SAT



$\exists$ x s.t. $C(x) = 1$?

$\exists$ x ?

$(OR \circ C)$-SAT



$\exists$ x s.t. $\exists\ i\ C_i(x) = 1$?

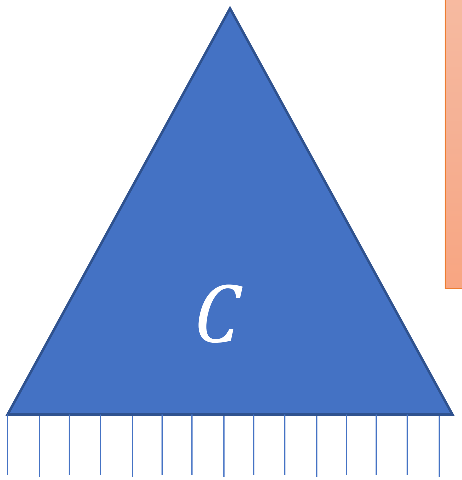
$\exists$ x ?

Motivation: New Structure Lemmas

What if we had a top $\mathbf{DOR}$ gate?

DOR:
Disjoint-OR
(at most one input is ever true)

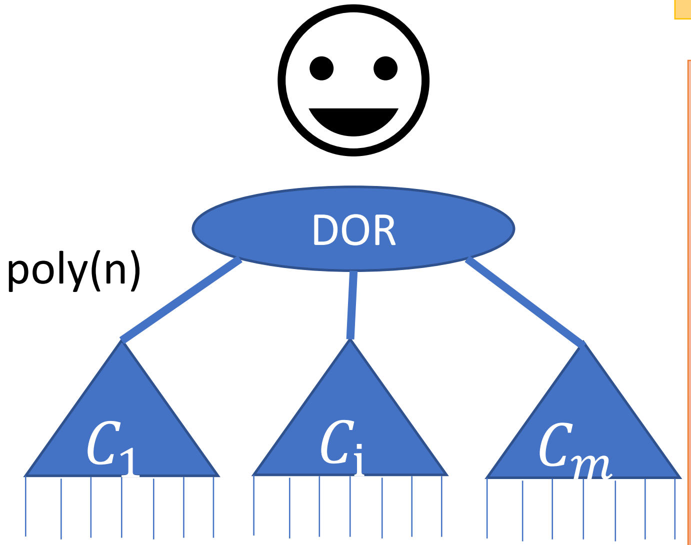
C-CAPP



Estimate
$\Pr_{x \sim U_n}[C(x) = 1]$,
with error ε

$x \sim U_n$

$(DOR \circ C)$-CAPP

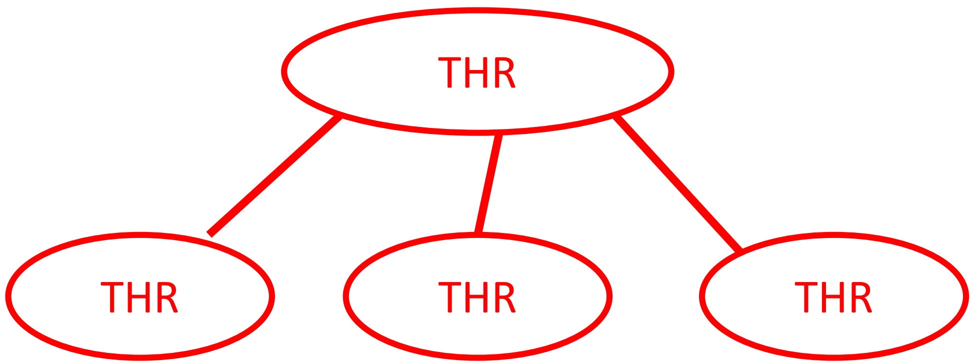


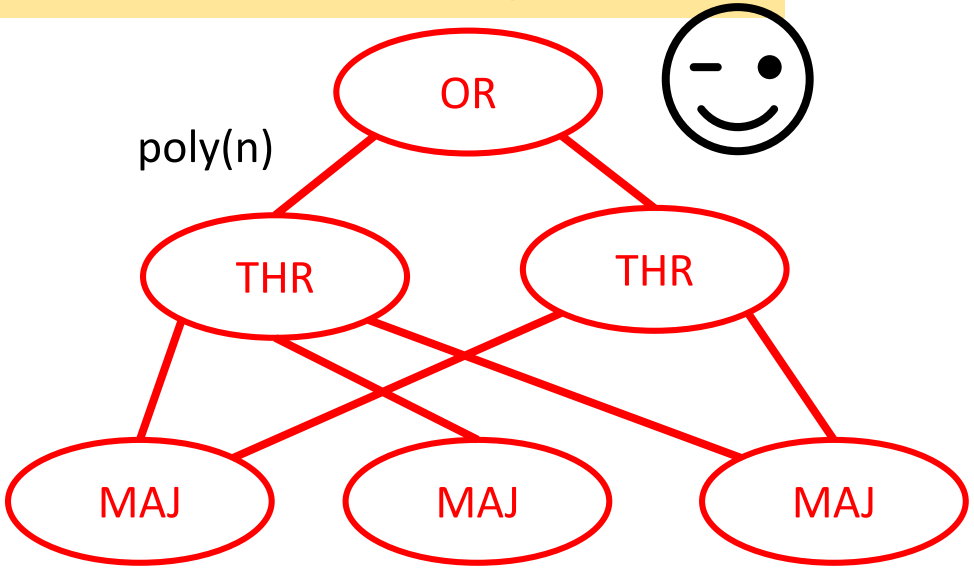
$x \sim U_n$

Estimate
$$\mathop{\mathrm{Ex}}_{x \sim U_n} \sum_i C_i(x).$$
$$= \sum_i \mathop{\mathrm{Ex}}_{x \sim U_n} [C_i(x)].$$

Structure Lemma I

$THR \circ THR$ can be explicitly written as an OR of poly-many $THR \circ MAJ$





Corollary: $2^n/n^{\omega(1)}$ time SAT for $THR \circ MAJ$ of poly size
$\Rightarrow 2^n/n^{\omega(1)}$ time SAT for $THR \circ THR$ of poly size

Structure Lemma II

For all $\varepsilon > 0$, every $THR \circ THR$ of size s with n inputs can be explicitly written as a $DOR \circ MAJ \circ MAJ$ circuit such that

(1): The DOR gate has $2^{\varepsilon n}$ fan-in.

(2): All $MAJ \circ MAJ$ sub-circuits have size $s^{O(1/\varepsilon)}$.

Two concrete settings:

1. $THR \circ THR \subseteq$ sub-exp DOR of $MAJ \circ MAJ$.
2. $THR \circ THR \subseteq$ poly DOR of sub-exp-size $MAJ \circ MAJ$.

Other Applications

For all $\varepsilon > 0$, every $THR \circ AND_k$ of size s with n inputs can be explicitly written as a $DOR \circ MAJ \circ AND_{2k}$ circuits, such that

(1): The DOR gate has $2^{\varepsilon n}$ fan-in.

(2): All $MAJ \circ AND_{2k}$ sub-circuits have size $s^{O(1/\varepsilon)}$.

Corollary:

A Polynomial Threshold Function (PTF) of degree k can be written as a sub-exp Disjoint-OR of PTF with poly-weights of degree $2k$.

Open Question 1

Can every $THR \circ THR$ be expressed as an OR of poly-many $MAJ \circ MAJ$?

This will show they are equivalent for non-trivial SAT algorithms.

Theorem:
Non-trivial #SAT algorithm for $MAJ \circ MAJ$
$\Rightarrow$ Non-trivial nondeterministic UNSAT algorithm for $THR \circ THR$.
(Enough for an NEXP lower bound against $THR \circ THR$.)

SAT for Depth-*d* THR ⇒ Depth-*d* Lower Bounds

The Depth Increase Issue [Ben-Sasson--Viola'14]:

$2^n/n^{\omega(1)}$ SAT algorithm for $\mathrm{AND}_3 \circ \boldsymbol{C}$

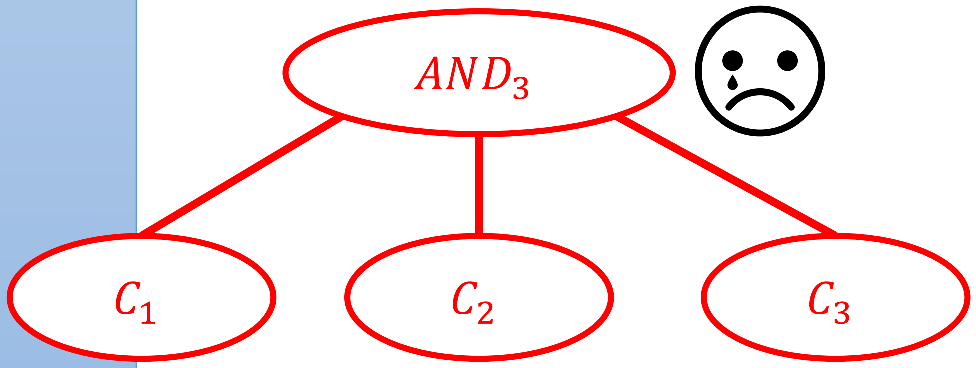
⇒ NEXP is not in $\boldsymbol{C}$.

To show SAT algs for THR of THR imply *analogous* lower bounds, we can't have this +1 increase in the depth ☹

Problem: We don't know whether

$$AND_3 \circ THR \circ THR \subseteq THR \circ THR$$

(maybe not?!)



Dealing With Depth Increase

We can use ETHR gates

ETHR gates : $f(x) = [w \cdot x = t]$, for some $w \in Z^n, t \in Z$.

[Hansen-Podolskii'10] proved some structural results:

$$THR_m \subseteq DOR_{poly(m)} \circ ETHR_m$$

$$ETHR \circ THR \subseteq THR \circ THR$$

$$AND \circ ETHR \subseteq ETHR$$

Dealing With Depth Increase

$THR_m \subseteq DOR_{poly(m)} \circ ETHR_m$

$AND_3 \circ THR \circ THR$

$AND_3 \circ DOR_{poly(n)} \circ ETHR \circ THR$

DOR = +
AND = ×

$DOR_{poly(n)} \circ AND_3 \circ ETHR \circ THR$

$AND \circ ETHR \subseteq ETHR$

$DOR_{poly(n)} \circ ETHR \circ THR$

$ETHR \circ THR \subseteq THR \circ THR$

$DOR_{poly(n)} \circ THR \circ THR$

Theorem: Non-trivial SAT/CAPP for $THR \circ THR$
⇒ Non-trivial SAT/CAPP for $AND_3 \circ THR \circ THR$
⇒ NEXP not in $THR \circ THR$

Open Question 2

Corollary:
If $(MAJ \circ MAJ)$-CAPP has a non-trivial algorithm,
Then NEXP not in $THR \circ THR$.

Can we "mine" any non-trivial SAT or CAPP algorithms from the ***exponential*** lower bound proofs for $MAJ \circ MAJ$ or $THR \circ MAJ$?

Would imply NEXP not in $THR \circ THR$!

Connection to Fine-Grained Complexity

$NEXP$ is not in $THR \circ THR$ would follow from "shaving logs" for several natural questions in **computational geometry**.

1. *Biochromatic Closest Pair Problem:*
   Given n red-blue points in polylog(n) dimensional Euclidean space, find the red-blue pair with minimum distance.
2. *Furthest Pair Problem:*
   Given n points in polylog(n) dimensional Euclidean space, find the pair with largest distance.
3. *Hopcroft's Problem:*
   Given n points and n hyperplanes in polylog(n) dimensional Euclidean space, is some point on some hyperplane?

If there is an $n^2/\log^{\omega(1)} n$ time algorithm for ***any*** of them
Then NEXP is not in $THR \circ THR$

A Simple CAPP-like Problem

$Apx\#MaxIP_{n,d}$: Given $A, B \subseteq \{0,1\}^d$ of size n and an integer t, approximate $\Pr_{(a,b)\in A\times B}[\langle a, b\rangle \geq t]$ with additive error ε.

By a simple reduction

Corollary:

If $Apx\#MaxIP_{n,d}$ for $d = polylog(n)$ can be solved with $\varepsilon = 1/polylog(n)$ in $n^2/\log^{\omega(1)} n$ time, then NEXP is not in $THR \circ THR$.

Open Question 3

Slightly improve the complexity of these computational geometry problems.
(Or show why they are unlikely?)

Would imply $NEXP$ is not in $THR \circ THR$.

Another Interesting Connection

k-SAT: Best Running Time is $2^{n(1-1/\Theta(k))}$.

Best Known L.B. for general TC_0 circuits:

[Impagliazzo-Paturi-Saks'93, Chen-Santhanam-Srinivasan'16]:

Parity requires $n^{1+c^{-d}}$ wires for depth-d TC_0 circuits.

[C.-Tell 18]:

There's a good reason for the $n^{1+\exp(-d)}$ bound!

It is consistent with the current state of knowledge that E^{NP} has TC_0 circuits of $O(\log\log n)$ depth and $O(n)$ wires.

Another Interesting Connection

Theorem:

An $2^{n(1-1/k^{1/\omega(\log\log k)})}$ time k-SAT algorithm

$\Rightarrow$

E^{NP} has no $O(n)$ size, $O(\log\log n)$-depth TC_0 circuits.

Based on the reduction from TC_0-SAT to k-SAT in [Abboud-Bringmann-Dell-Nederlof’18].

Conclusion

1. Poly-size $THR \circ THR$ and $THR \circ MAJ$ are **equivalent** for Non-Trivial SAT Algorithms
2. Poly-size $THR \circ THR$ and $MAJ \circ MAJ$ are **equivalent** for Non-Trivial CAPP Algorithms!
3. Non-trivial SAT algorithms for $THR \circ MAJ$ or CAPP algorithms for $MAJ \circ MAJ$ $\Rightarrow$ NEXP not in $THR \circ THR$. *(No depth Increase)*
4. **Slightly** improving the running times on geometry problems $\Rightarrow$ NEXP not in $THR \circ THR$.

Open Question

1. Can every $THR \circ THR$ be expressed as an OR of poly-many $MAJ \circ MAJ$?
2. Can we “mine” any non-trivial SAT or CAPP algorithms from the ***exponential*** lower bound proofs for $MAJ \circ MAJ$ or $THR \circ MAJ$?
3. ***Slightly*** improve the complexity of these computational geometry problems. *(Or show why it is unlikely?)*

Thanks