Lower Bounds for Unrestricted Boolean Circuits: Open Problems

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Simons Institute, September 10, 2018

Unrestricted Boolean Binary Circuits

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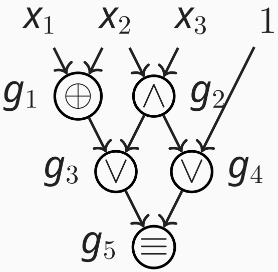

Fundamental Question

Given a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, what is the minimum number of gates needed to compute f?

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Does there exist an infinite sequence of functions $f_1, f_2, \ldots$ such that f_n has n inputs, $\bigcup_{n=1}^{\infty} f_n^{-1}(1) \in \mathrm{NP}$, and f_n requires superpoly(n) gates? (This would mean that $\mathrm{P} \neq \mathrm{NP}$)

Exponential Bounds

Lower Bound

Counting shows that almost all functions of n variables have circuit size $\Omega(2^n/n)$ [S49]

Upper Bound

Any function can be computed by circuits of size $(1 + o(1))2^n/n$ [L58]

Explicit Lower Bounds

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What can we prove for explicit functions?

Outline

1. Gate Elimination
2. Multi-Output Functions
3. Non-Gate-Elimination Lower Bounds
4. Symmetric Functions
5. Satisfiability Algorithms
6. Mass Production
7. Logarithmic Depth Circuits

Outline

1. Gate Elimination

   How to prove, say, a $3n$ lower bound for a Boolean function f?

2. Multi-Output Functions
3. Non-Gate-Elimination Lower Bounds
4. Symmetric Functions
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Gate Elimination Method

- Show that f is resistant to about n substitutions
- Show that one can always find a substitution eliminating at least 3 gates

Lower Bounds

- The currently best known lower bound $\left(3 + \frac{1}{86}\right) n$ is proved by gate elimination [FGHK16]
- The corresponding function f is affine disperser for sublinear dimension: f is non-constant on any affine subspace of $\{0, 1\}^n$ of large enough dimension
- Explicit constructions of such functions were found relatively recently [BK12]

Linear Size Circuits for Affine Dispersers

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Open problem: Do there exist affine dispersers for sublinear dimension of linear circuit size?

Quadratic Dispersers

Open problem: Construct an explicit "quadratic" disperser *f* (even in NP, even with $o(n)$ outputs) that is not constant on any set $S \subseteq \{0,1\}^n$ of size at least $2^{n/100}$ that can be defined as

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This will give an improved lower bound (about $3.1n$) [GK16]

Limitations of Gate Elimination

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- Formally, there exist circuits such that any substitution of the form $x \leftarrow g$, where g is an arbitrary function, removes no more than five gates from the circuit [GHKK16]. Therefore, one definitely needs new ideas to get something stronger than $5n$

Outline

1. Gate Elimination
2. Multi-Output Functions

   Can one prove stronger lower bounds for functions with multiple outputs?

3. Non-Gate-Elimination Lower Bounds
4. Symmetric Functions
5. Satisfiability Algorithms
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Open problem: How to prove a $5n$ lower bound for an n-to-n function?

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1. Gate Elimination
2. Multi-Output Functions
3. Non-Gate-Elimination Lower Bounds

   Are there approaches other than gate elimination for proving lower bounds for unrestricted circuits?

4. Symmetric Functions
5. Satisfiability Algorithms
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Open problem: Can any of these non-gate-elimination methods be extended to get stronger than $2n$ lower bounds?

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1. Gate Elimination
2. Multi-Output Functions
3. Non-Gate-Elimination Lower Bounds
4. Symmetric Functions

   Can one prove a superlinear lower bound for a symmetric function?

5. Satisfiability Algorithms
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Symmetric Functions

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Open problem: What is $C(\mathrm{SUM}_n)$?

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5. Satisfiability Algorithms

    Given a circuit, how hard is it to find an assignment making this circuit to output 1?

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Open problem: Do non-trivial satisfiability algorithms for circuits of size cn imply cn circuit lower bounds?

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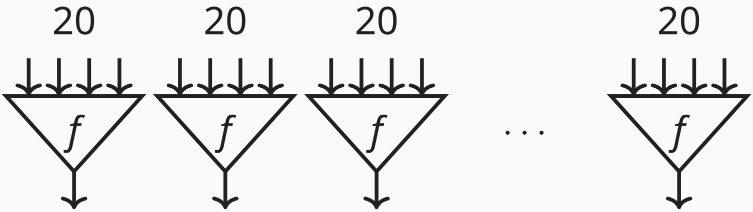
- It is natural to expect that an optimal circuit for g looks as follows:



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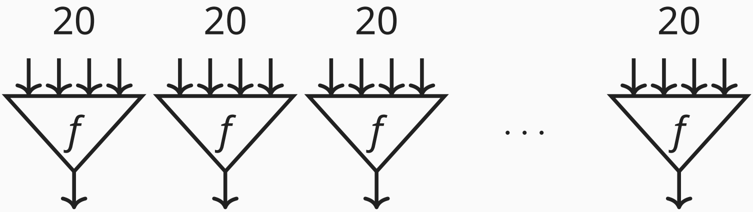

- But we don't know how to prove this!

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- At the same time, it does occur for very hard functions: if $C(g) \approx 2^n/n$, then $C(g, g) \approx C(g)$ [U74]

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Open problem: What are the functions avoiding mass production effect?

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   Can we at least prove superlinear lower bounds on circuits of logarithmic depth?

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Open problem: Improve $2^{\sqrt{n}}$ lower bound for depth three circuits.

Depth Three Circuits

- Lower bounds of the form $2^{n/k}$ are known for OR $\circ$ AND $\circ$ OR$_k$ circuits (i.e., OR of k-CNFs) [PPZ97]

- For $k = 2$, a lower bound $2^{n-o(n)}$ is known [PSZ00]

Converting Small Size Circuits into Non-trivial Depth 3 Formulas

Theorem [V77]

For any circuit of size $O(n)$ and depth $O(\log n)$, there exists an

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Theorem

For any circuit of size s and any depth, there exists an

$$\mathrm{OR}_{2^{\frac{s}{2.5}}} \circ \mathrm{AND} \circ \mathrm{OR}_{16}$$

formula computing the same function.

Open Problem

Open problem: Can one convert a circuit with s gates into a, say,

$$\mathrm{OR}_{2^{\frac{s}{4}}} \circ \mathrm{AND} \circ \mathrm{OR}_2$$

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formula? More generally, is is true that any circuit of size cn can be converted into a

$$\mathrm{OR}_{2^{(1-\varepsilon(c))n}} \circ \mathrm{AND} \circ \mathrm{OR}_{\delta(c)}$$

formula?

Summary of Open Problems

1. Prove that there exists an affine disperser of linear circuit size!
2. Construct an explicit quadratic disperser!
3. Prove a $5n$ lower bound for an n-to-n function!
4. Prove $3n$ lower bound without gate elimination!
5. Find $C(\mathrm{SUM}_n)$!
6. Prove that faster than brute force SAT algorithm for circuits of size cn imply cn circuit lower bounds!
7. Construct functions avoiding mass production effect!
8. Convert lower bounds for depth-3 circuits to lower bounds for unrestricted circuits!

Thank you!