Undecidability of Linear Inequalities Between Graph Homomorphism Densities

Hamed Hatami
joint work with Sergey Norin

School of Computer Science
McGill University

December 4, 2013

Introduction

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- Discovery of rich algebraic structure underlying many of these techniques.
- Neater proofs with no low-order terms.
- Methods for applying these techniques in semi-automatic ways.

Density of H in G

$$\frac{\text{number of copies of } H \text{ in } G}{\binom{|V(G)|}{|V(H)|}}.$$

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- Many fundamental theorems in extremal graph theory can be expressed as algebraic inequalities between subgraph densities.

Theorem (Freedman, Lovász, Schrijver 2007)

Every such inequality follows from the positive semi-definiteness of a certain infinite matrix.

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Razborov's flag algebras

A formal calculus capturing many standard arguments (induction, Cauchy-Schwarz,...) in the area.

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- HH, Hladky, Kral, Norin, Razborov: A question of Sidorenko and Jagger, Šťovíček and Thomason.
- HH, Hladky, Kral, Norin, Razborov: A conjecture of Erdös.

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- Razborov: Minimal density of triangles, given an edge density.
- Razborov: Turán's hypergraph problem under mild extra conditions.
- other conjectures of Erdös, crossing number of complete bipartite graphs, etc.

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Can every true algebraic inequality between subgraph densities be proved using a finite amount of manipulation with subgraph densities of finitely many graphs?

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Can every true algebraic inequality between subgraph densities be proved using a finite amount of manipulation with subgraph densities of finitely many graphs?

HH-Norine 2011

The answer is negative in a strong sense.

Formal definitions

Extremal graph theory

Studies the relations between the number of occurrences of different subgraphs in a graph G.

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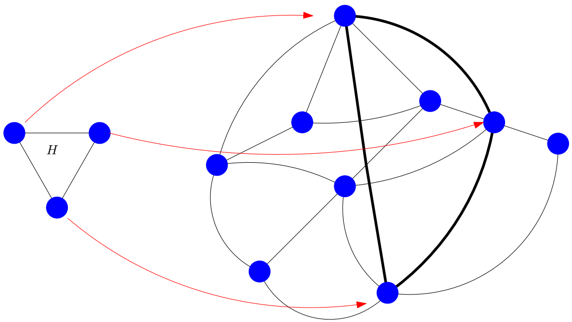
Equivalently one can study the relations between the "homomorphism densities".

Homomorphism Density

Definition

- Map the vertices of H to the vertices of G independently at random.



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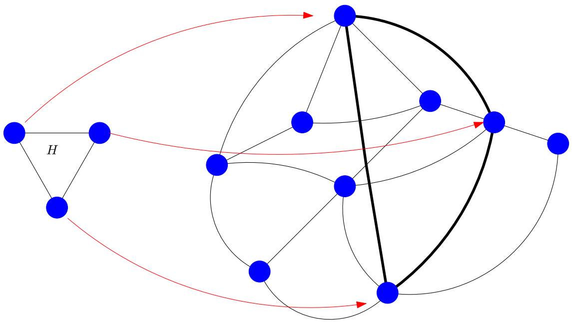
$$t_H(G) := \Pr[\text{edges go to edges}].$$



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A map $f : H \to G$ is called a homomorphism if it maps edges to edges.

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- Asymptotically $t_H(\cdot)$ and subgraph densities are equivalent.

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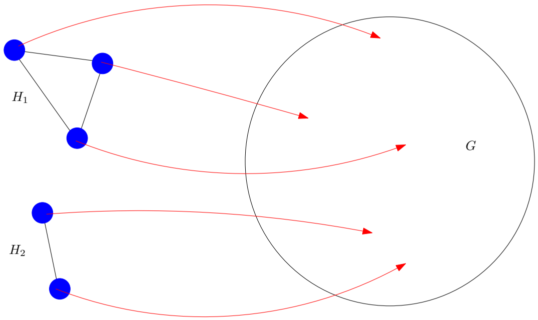
- The functions t_H have nice algebraic structures:

$$t_{H_1 \sqcup H_2}(G) = t_{H_1}(G) t_{H_2}(G).$$



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Example (Goodman's bound 1959)

$$t_{K_3}(G) - 2t_{K_2 \sqcup K_2}(G) + t_{K_2}(G) \geq 0.$$

Algebra of Partially labeled graphs

Definition

A partially labeled graph is a graph in which some vertices are labeled by *distinct* natural numbers.

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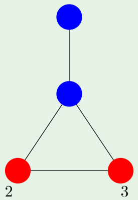

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Let H be partially labeled with labels L. For $\phi : L \to G$, define

$$t_{H,\phi}(G) := \Pr\left[f : H \to G \text{ is a hom.} \mid f|_L = \phi\right].$$

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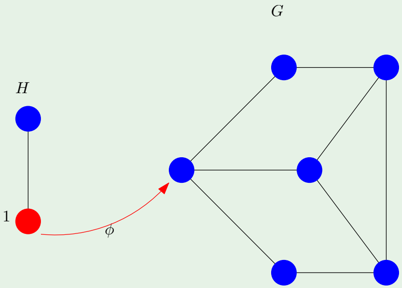

$$t_{H,\phi}(G) = \frac{3}{6} = \frac{1}{2}.$$

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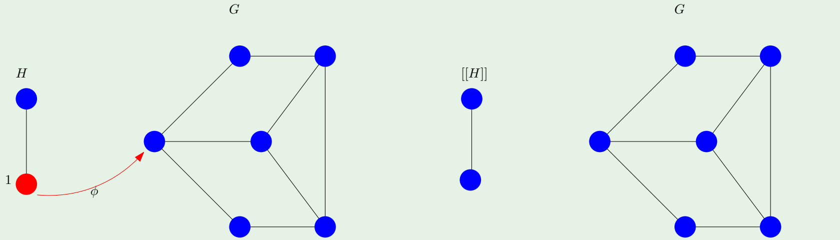

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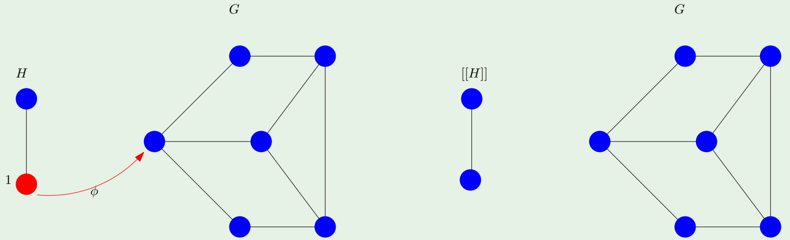

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$$\mathbb{E}_\phi \left[t_{H,\phi}(G) \right] = t_{[H]}(G)$$

- Recall that:

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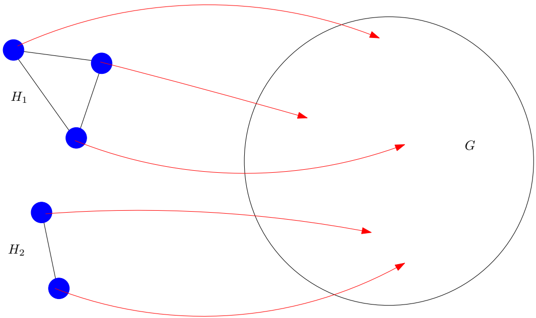

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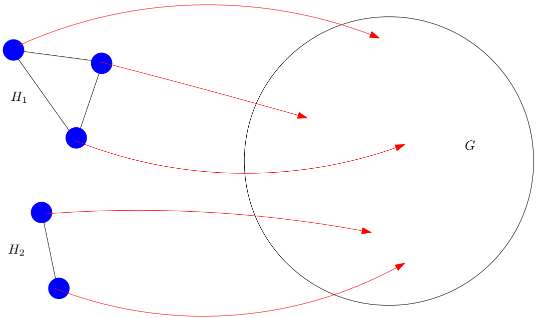

- This motivates us to define $H_1 \times H_2 := H_1 \sqcup H_2$.

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- If multiple edges arise, only one copy is kept.

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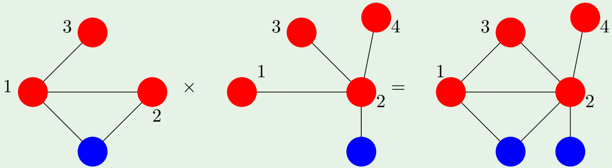

- Let H_1 and H_2 be partially labeled with labels L_1 and L_2.

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- We have $t_{H_1,\phi}(G) t_{H_2,\phi}(G) = t_{H_1 \times H_2,\phi}(G)$.

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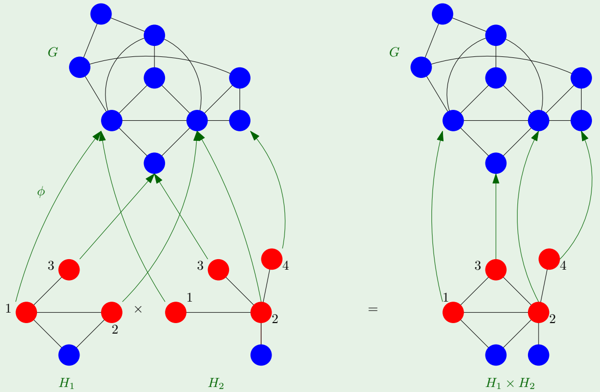

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$$
\begin{aligned}
0 &\leq \left(\sum b_i t_{H_i,\phi}(G)\right)^2 = \sum b_i b_j t_{H_i,\phi}(G) t_{H_j,\phi}(G) \\
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- Condition III: The infinite matrix whose ij-th entry is $t_{[H_i \times H_j]}(G)$ is positive semi-definite.

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Theorem (Freedman, Lovász, Shrijver 2007)

These conditions describe the closure of the set

$$\{(t_{F_1}(G), t_{F_2}(G), \ldots) \ : \ G\} \in [0,1]^{\mathbb{N}}.$$

Quantum Graphs

- A quantum graph is a formal linear combination of graphs:

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- A quantum graph $a_1 H_1 + \ldots + a_k H_k$ is called positive, if for all G,

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- Partially labeled quantum graphs form an algebra:

$$(a_1 H_1 + \ldots + a_k H_k) \cdot (b_1 L_1 + \ldots + b_\ell L_\ell) = \sum a_i b_j H_i \cdot L_j.$$

Unlabeling operator

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Corollary

Always

$$\left[g_1^2 + \ldots + g_k^2\right] \geq 0.$$

Question (Lovász's 17th Problem, Lovász-Szegedy, Razborov)

Is it true that every $f \geq 0$ is of the form

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Observation (Lovasz-Szegedy and Razborov)

If $f \geq 0$ and $\epsilon > 0$, there exists a positive integer k and quantum labeled graphs $g_1, g_2, \ldots, g_k$ such that

$$-\epsilon \leq f - \left[g_1^2 + g_2^2 + \ldots + g_k^2\right] \leq \epsilon.$$

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Theorem (HH and Norin)

The answer to the above question is negative.

positive polynomials

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Theorem (Hilbert 1888)

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Example (Motzkin's polynomial)

$$x^4y^2 + y^4z^2 + z^4x^2 - 6x^2y^2z^2 \geq 0.$$

Extending to quantum graphs

Theorem (HH and Norin)

There are positive quantum graphs f which are not sums of squares. That is, always $f \neq \left[g_1^2 + \ldots + g_k^2\right]$.

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- The proof is based on converting $x^4y^2 + y^4z^2 + z^4x^2 - 6x^2y^2z^2$ to a quantum graph.

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- Co-recursively enumerable: Try to find a point that makes p negative.
- recursively enumerable: Try to write $p = \sum (p_i/q_i)^2$.

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- Maybe at least the decidability? (A 10th problem)

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The following problem is undecidable.

- QUESTION: *Does the inequality $a_1 t_{H_1}(G) + \ldots + a_k t_{H_k}(G) \geq 0$ hold for every graph G?*

Proof

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Equivalently

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Instead I will prove the following theorem:

Theorem

The following problem is undecidable.

- INSTANCE: *A polynomial* $p(x_1, \ldots, x_k, y_1, \ldots, y_k)$.
- QUESTION: *Does the inequality* $p(t_{K_2}(G_1), \ldots, t_{K_2}(G_k), t_{K_3}(G_1), \ldots, t_{K_3}(G_k)) \geq 0$ *hold for every* $G_1, \ldots, G_k$*?*

- Matiyasevich 1970 Solution to Hilbert's 10th problem: Checking the positivity of $p \in \mathbb{R}[x_1, \ldots, x_k]$ on $\left\{1 - \frac{1}{n} : n \in \mathbb{Z}\right\}^k$ is undecidable.

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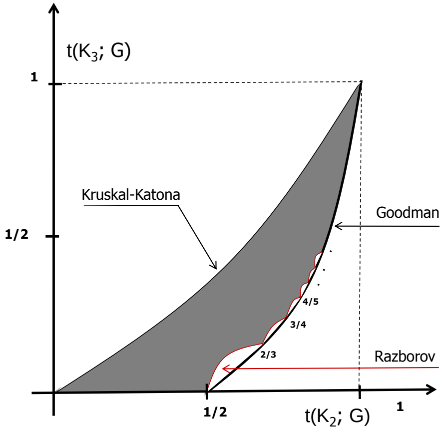
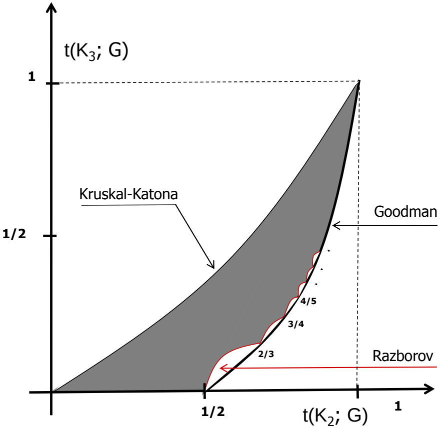
- Bollobás, Razborov: Goodman's bound is achieved only when $t_{K_2}(G) \in \left\{1 - \frac{1}{n} : n \in \mathbb{Z}\right\}$.



Let S be the grey area and $g(x) = 2x^2 - x$. (Goodman: $t_{K_3}(G) \geq 2t_{K_2}(G)^2 - t_{K_2}(G)$.)



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Lemma

Let $p \in \mathbb{R}[x_1, \ldots, x_k]$. Define $q(x_1, \ldots, x_k, y_1, \ldots, y_k)$ as

$$q := p \prod_{i=1}^{k} (1 - x_i)^6 + C_p \times \left(\sum_{i=1}^{k} y_i - g(x_i) \right).$$

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- $q < 0$ *for some* $x_i = t_{K_2}(G_i)$ *and* $y_i = t_{K_3}(G_i)$. *(reduction)*

Where do we go from here?

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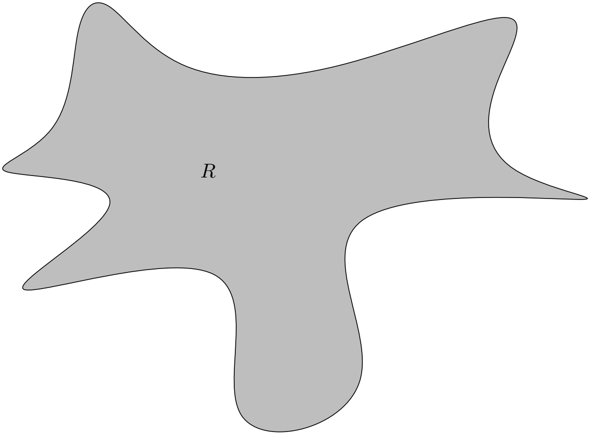
- Question: What about unions of cliques?

- Let R denote the closure of $\{(t_{H_1}(G), t_{H_2}(G), \dots) \ : \ G\} \subset [0,1]^{\mathbb{N}}$.



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[Glebov, Klimošová, Král 2013+]

There are finitely forcible W's such that $\{W(x,\cdot) : x \in [0,1]\}$ with the L_1 distance contains a subset homeomorphic to $[0,1]^\infty$.